Primer on Complex Numbers

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1 Introduction

Much of the analysis involved in understanding quantum computation depends on our ability to manipulate complex numbers, and so it is very important that you spend some time familiarising yourself with them. These notes are intended as a short primer to help you get started. If you have studied complex numbers recently (during your degrees), then you probably won't need these notes. If you last thought about complex numbers during your A-levels (or equivalent), then hopefully these notes will be a useful revision aid (and I urge you to revise them—you will need to be quite fluent in their manipulation). If you haven't studied complex numbers before, then you're probably going to have to put some work in. I'm afraid this is unavoidable, as without complex numbers we simply cannot study some of the most important aspects of quantum computing.

There is a problem sheet accompanying this primer which will give you some practice in manipulating complex numbers, if you need it. If this is truly the first time you've encountered them, then you'll probably need some more problems to really get the hang of it. I would strongly recommend you get hold of a copy of the texbook

Engineering Mathematics, by K.A. Stroud (MacMillan Press, 1995). Available in the main library under classmark TA345.

There are both short- and long-loan copies available. There is also an accompanying volume: *Engineering mathematics : programmes and problems*, by the same author, available under the same classmark.

2 What are complex numbers and why do we need them?

The very phrase "complex number" implies something that is complicated. This is not the case! At the level we require, complex numbers are really quite simple.

Let us revisit some basic arithmetic for a moment. Recall the definition of a square root: if $y = x \times x \equiv x^2$ (y equals x-squared), then equally, $x = \pm\sqrt{y}$ (x equals plus-or-minus the square-root of y). Notice that x can take either a $+$ or $-$ sign, since $(-x)\times(-x)=x^2$, a double negative. Some examples:

$$
\begin{aligned}
y=1 &\quad\mapsto\quad \sqrt{y}=\pm 1, &(1)\\
y=4 &\quad\mapsto\quad \sqrt{y}=\pm 2, &(2)\\
y=7 &\quad\mapsto\quad \sqrt{y}=\pm 2.64575\ldots\quad . &(3)
\end{aligned}
$$

What happens if $y<0$? Can you think of a real number whose square is negative? There isn't one: real numbers are either positive or negative, and so taking the square of a real always gives a postive result. This means that we don't know how to solve a simple equation such as

$$x^2+1=0 \qquad \mapsto x=?. \tag{4}$$

Just as integers are insufficient for us to solve $x^2-3=0$, so real numbers are insufficient to allow us to solve $x^2+1=0$. We need something else. That something is called an *imaginary number*, which we will call i, and is defined by

$$\mathrm{i}^2=-1 \quad\leftrightarrow\quad \mathrm{i}=\sqrt{-1} \tag{5}$$

Hence the solution to $x^2+1=0$ is $x=\pm\mathrm{i}$. This allows us to calculate the square root of any negative real number. Noting that $\sqrt{xy}=\sqrt{x}\sqrt{y}$, we have $\sqrt{-y}=\sqrt{-1}\sqrt{y}=\pm\mathrm{i}\sqrt{y}$. Some examples:

$$
\begin{aligned}
y=-1 &\quad\mapsto\quad \sqrt{y}=\pm\mathrm{i}, &(6)\\
y=-4 &\quad\mapsto\quad \sqrt{y}=\pm 2\mathrm{i}, &(7)\\
y=-7 &\quad\mapsto\quad \sqrt{y}=\pm 2.64575\ldots\mathrm{i}. &(8)
\end{aligned}
$$

Let us note some properties of i (which is often written as j in some disciplines). In particular, what happens if we raise i to different powers?

$$
\begin{aligned}
\mathrm{i} &= \sqrt{-1} &(9)\\
\mathrm{i}^2 &= -1 &(10)\\
\mathrm{i}^3 &= \mathrm{i}^2\mathrm{i}=-\mathrm{i} &(11)\\
\mathrm{i}^4 &= \mathrm{i}^2\mathrm{i}^2=(-1)(-1)=1. &(12)
\end{aligned}
$$

It is then very easy to evaluate higher powers of i: $\mathrm{i}^7=\mathrm{i}^4\mathrm{i}^3=-\mathrm{i}$; $\mathrm{i}^{10}=\mathrm{i}^4\mathrm{i}^4\mathrm{i}^2=-1$ etc.

So, is it enough to have both real and imaginary numbers? It turns out that there are some problems which require us to have both together! Consider the general quadratic equation $y(x)=ax^2+bx+c$, where a, b and c are constants. The solutions to the equation $y(x)=0$ is given by $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Note that there are two solutions ($\pm$). Let's look at a couple of examples:

$$y(x)=x^2-4=0 \qquad\qquad \mapsto x=\pm 2, \tag{13}$$

the solutions are purely real.

$$y(x) = x^2 + 9 = 0 \qquad \mapsto x = \pm 3\mathrm{i}, \tag{14}$$

the solutions are purely imaginary.

$$y(x) = x^2 - x + 1 = 0 \qquad \mapsto x = \frac{1 \pm \mathrm{i}\sqrt{3}}{2} \tag{15}$$

the solutions have both real and imaginary parts. Since real and imaginary numbers are fundamentally different, such a number is neither real, nor is it imaginary. It is a *complex* number.

A complex number is nothing more than an ordered pair of real numbers (a, b), which are usually written as a sum of real and imaginary parts,

$$z = a + \mathrm{i}b. \tag{16}$$

The real number a is called the real part of z, $a = \mathrm{Re}\,(z)$; and the real number b is called the imaginary part of z, $b = \mathrm{Im}\,(z)$. Note that $\mathrm{Im}\,(z) \neq \mathrm{i}b$. Real and imaginary numbers are subsets of the complex numbers with $a = 0$ for the set of all imaginary numbers, and $b = 0$ for the set of all real numbers.

3 Manipulating Complex Numbers

We can perform the usual arithmetic operations on complex numbers. The rules are as follows.

3.1 Addition and Subtraction

This is very easy. If $z_1 = a + \mathrm{i}b$ and $z_2 = c + \mathrm{i}d$, then $z_1 + z_2 = (a + c) + \mathrm{i}(b + d)$. Clearly $\mathrm{Re}\,(z_1 + z_2) = \mathrm{Re}\,(z_1) + \mathrm{Re}\,(z_2)$, and similarly for the imaginary part. For example, if $z_1 = 3 + 4\mathrm{i}$ and $z_2 = -4 + 2\mathrm{i}$, then $z_1 + z_2 = (3 - 4) + (4 + 2)\mathrm{i} = -1 + 6\mathrm{i}$. Notice that two complex number can add to give a pure real number (if $b = -d$) or a pure imaginary number (if $a = -c$).

3.2 Multiplication of two complex numbers

Multiplying two complex numbers is simply a matter of combining all the possible terms, and then simplifying the result. If $z_1 = a + \mathrm{i}b$ and $z_2 = c + \mathrm{i}d$, then the product $z_1 z_2 = ac + \mathrm{i}ad + \mathrm{i}bc + \mathrm{i}^2 bd$. Remembering that $\mathrm{i}^2 = -1$, this simplifies to $z_1 z_2 = (ac - bd) + \mathrm{i}(ad + bc)$. Example: $(4+5\mathrm{i})(1+2\mathrm{i}) = (4\times1-5\times2)+\mathrm{i}(4\times2+5\times1) = -6+13\mathrm{i}$. Notice that multiplying by a pure real number simply multiplies both real and imaginary parts: $c(a+\mathrm{i}b) = ac+\mathrm{i}bc$.

In general, the multiplication of two complex numbers yields a third complex number. There is a special case where it can yield a real number. When $z_1 = a + ib$ and $z_2 = a - \mathrm{i}b$, the product $z_1 z_2 = a^2 + b^2$ is a purely real number. We say that z_2 is the *complex conjugate* of z_1, written as $z_2 = z_1^*$. We will learn more about this later.

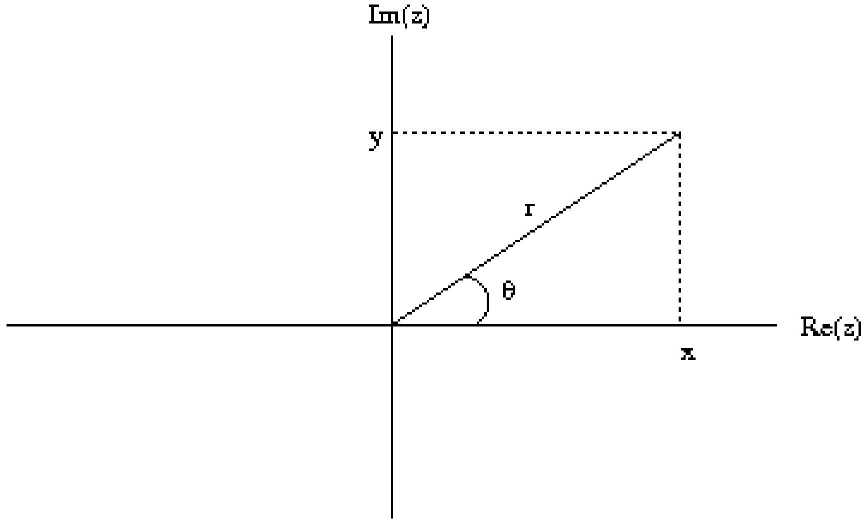

Figure 1: Crossing the real and imaginary number lines forms the complex plane, illustrated using an Argand diagram.

3.3 Division of two complex numbers

This is a little trickier than multiplication. Say we have two complex numbers z_1, z_2, and we wish to find the ratio $\frac{z_1}{z_2}$. It's not obvious how to do this. The way to proceed is as follows. We multiply both numerator and denominator by z_2^*, the complex conjugate of z_2: $\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*}$. The denominator is now purely real, and we can simply divide the real and imaginary parts of the numerator individually. Example: $z_1 = 1 + \mathrm{i}$, $z_2 = 2 - \mathrm{i}$. We want to find $\frac{z_1}{z_2} = \frac{1+\mathrm{i}}{2-\mathrm{i}}$. The complex conjugate of z_2 is $z_2^* = 2 + \mathrm{i}$, and we multiply both numerator and denominator: $\frac{z_1}{z_2} = \frac{(1+\mathrm{i})(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})}$. We then evaluate both numerator and denominator to give $\frac{z_1}{z_2} = \frac{1+3\mathrm{i}}{5} = \frac{1}{5} + \frac{3}{5}\mathrm{i}$.

4 An Alternative Representation

There is a useful pictorial way to think about complex numbers. We can represent real numbers on a number line, and similarly with imaginary numbers. Complex numbers can be represented on the plane formed by crossing these two number lines, as shown in figure 1. This type of diagram is often called an *Argand diagram*.

It should be fairly clear that any complex number can be represented by a point on the Argand Diagram, with purely real numbers (1, 3, 7,-0.345, etc) lying on the horizontal axis, and imaginary numbers (i, 2.3i, −6i) lying on the vertical axis. For a and b both positive, we note that $a + \mathrm{i}b$ lies in the upper right quadrant, $a - \mathrm{i}b$ is in the lower right quadrant, $-a + \mathrm{i}b$ is in the upper left quadrant, and $-a - \mathrm{i}b$ is in the lower left quadrant. Note that the complex conjugate of a complex number is simply a reflection across the real (horizontal) axis (since only the sign of the imaginary part changes).

This diagrammatic representation of complex numbers suggests an alternative algrabraic form that is very useful. Our current algrebraic representation ($z = x + \mathrm{i}y$) uses the fact that we can uniquely specify a complex number by its position on the horizontal and vertical axes. There is another unique specification which we often use, often called the *polar* form. This relies on us being able to represent points in the the complex plane by the distance they are from the origin (r in the diagram), and the angle they make with the real axis (θ). We can write down the following relations between the cartesian and polar coordinates:

$$x = r\cos\theta; \quad y = r\sin\theta; \quad x^2 + y^2 = r^2; \quad \theta = \arctan\tfrac{y}{x}$$
$$\mapsto \qquad z = x + \mathrm{i}y \equiv r(\cos\theta + \mathrm{i}\sin\theta)$$

It can also be shown (but this is beyond the scope of this primer) that this can be rewritten in the very useful polar form

$$r(\cos\theta + \mathrm{i}\sin\theta) \equiv r\mathrm{e}^{\mathrm{i}\theta} \tag{17}$$

The ordered pair (r, θ) are known as the modulus and argument of the complex number. It should be quite obvious that a positive real number c has $r = c$ and $\theta = 0$. It's negative counterpart $-c$ has $r = c$ and $\theta = \pi$ (or any integer multiple thereof. Purely imaginary numbers have $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ etc.

4.1 Multiplication and Division of Complex Numbers in the Polar Form

A major benefit of the polar representation is that multiplication and division are very easy (addition and subtraction are definitely best done in the cartesian form). For two complex numbers $z_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}$, $z_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2}$, the product of them is $z_1z_2 = r_1r_2\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$. The ratio of them is $\frac{z_1}{z_2} = \frac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)}$. Note that multiplying by a real number simply multiplies the modulus of the complex number by that scalar.

4.2 Powers of Complex Numbers

As well as making multiplication and division very easy, the polar form of complex numbers allows us to takes powers of complex numbers:

$$z^n = \left(r\mathrm{e}^{\mathrm{i}\theta}\right)^n = r^n\mathrm{e}^{\mathrm{i}n\theta}, \tag{18}$$

This allows us, for example, to calculate the square root of a complex number. This is actually quite a subtle calculation which is beyond the scope of these notes.

Finally, we can easily calculate the complex conjugate in the polar form. We'd already noted that complex conjugation simply reflects a complex number in the real axis. This is the same as negating the angle θ, and so

$$z = r\mathrm{e}^{\mathrm{i}\theta} \mapsto z^* = r\mathrm{e}^{-\mathrm{i}\theta}. \tag{19}$$

Note well that $z^*z = r^2$. This will be used frequently in this course.