Lecture 2: The Quantum Mechanics of Qubits

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Introduction

The essence of quantum mechanics is encapsulated in its four “postulates”. In this lecture we will:

- Introduce two of the four postulates of quantum mechanics, beginning with
  - The notion of a state, and the notation we use to denote states
  - The constraints that a state must obey
- and moving onto
  - What information is stored in a quantum state?
  - How do we access that information?

Reminder of Qubits

- Last lecture we introduced the qubit as a two-state quantum system
- The general qubit state is a superposition of states $|0\rangle$ and $|1\rangle$ and is written $|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$
- When we measure the state of a qubit, the result can only be 0 or 1
- The result of the measurement is probabilistic
- The rules that determine what a_0 and a_1 can be, how they can change, and what they mean are known as the **postulates of quantum mechanics**

The First Postulate

Every quantum system is described completely by a state vector. All properties of the system can be deduced from the state vector.

- Every quantum system exists in its own “state space”
- Quantum states are equivalent to positions in that space
- The state space is *complex*
- Positions in the space are described in terms of the **basis states**

State Notation

We use the so-called Dirac notation to describe quantum states

- A state is denoted by $|\psi\rangle$, which is also called a state **ket** or a state vector
- A state vector is a weighted sum of the basis states:

$$|\psi\rangle = a_0 |\phi_0\rangle + a_1 |\phi_1\rangle + \cdots + a_{N-1} |\phi_{N-1}\rangle = \sum_{i=0}^{N-1} a_i |\phi_i\rangle$$

- The basis states $\{|\phi_i\rangle\}$ allow us to describe any point in the state space of the system
- We will often write the state as a vector of the coefficients:

$$|\psi\rangle \equiv \begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \end{pmatrix}$$

More Notation

- There is also a conjugate state $\langle\psi|$, also called a state **BRA**
- If $|\psi\rangle = \sum_i a_i |\phi_i\rangle$, then $\langle\psi| = \sum_i a_i^* \langle\phi_i| = \left(a_0^*, a_1^*, \ldots, a_{N-1}^*\right)$
- Bra states are row vectors
- Basis vectors are the conjugates of the basis in ket-space
- Coefficients are the complex conjugates of the coefficients in ket space

Quantum Notation for Qubits

- For a qubit, there is a natural choice of basis: $\{|0\rangle, |1\rangle\}$ known as the *computational* or X basis
- The general state of a qubit is hence $|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$
- We'll often write this as a vector $|\psi\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$
- a_0 and a_1 may be **complex** numbers
- The corresponding bra state is $\langle\psi| = a_0^* \langle 0| + a_1^* \langle 1| = \left(a_0^*, a_1^*\right)$

Inner Products

- It is useful to know how "similar" two states are
- The means of doing this is known as the **inner product**
- It is the product of a bra and a ket state and is defined as

$$\langle\psi|\beta\rangle = \left(\sum_i a_i^* \langle\phi_i|\right)\left(\sum_j b_j |\phi_j\rangle\right) = \sum_{ij} a_i^* b_j \langle\phi_i|\phi_j\rangle$$

- We will only work with orthogonal, normal (orthonormal) bases which have the property $\langle\phi_i|\phi_j\rangle = \delta_{ij}$
- The inner product then becomes

$$\langle\psi|\beta\rangle = \sum_{ij} a_i^* b_j \langle\phi_i|\phi_j\rangle = \sum_i a_i^* b_i$$

Inner Products on Qubits

- For qubits, the basis has the properties $\langle 0|0\rangle = \langle 1|1\rangle = 1$, and $\langle 0|1\rangle = \langle 1|0\rangle = 0$: it's an orthonormal basis
- So for $|\psi_1\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$, $|\psi_2\rangle = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$:

$$\langle \psi_1|\psi_2\rangle = a_0^* b_0 + a_1^* b_1$$

- If $\langle \phi_1|\phi_2\rangle = 0$, the states are *orthogonal*
- If $\langle \psi|\psi\rangle = 1$, then $|\psi\rangle$ is said to be **normalised**
- For a quantum state to be valid it *must* be normalised

The Second Postulate

> The probability of a measurement on a quantum system giving a certain result is determined by the weight of the relevant basis state in the state vector. After the measurement, the system is in the state corresponding to the result of the measurement

- Consider $|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle$
- A measurement of the state can only yield **two** possible results: 0 or 1
- Outcome is probabilistic, with
- $p(0) = a_0^* a_0 = |a_0|^2$
- $p(1) = a_1^* a_1 = |a_1|^2$
- $p(0) + p(1) = a_0^* a_0 + a_1^* a_1 = 1$ must hold: normalisation

Post-measurement states

- After the measurement, the state of the system is the state associated with the result of the measurement
- If the result of the measurement 1, then the state of the qubit after the measurement is $|1\rangle$

Example: A qubit is in state $|\psi\rangle = \frac{1}{\sqrt{3}}\,|0\rangle + \sqrt{\frac{2}{3}}\,|1\rangle$.

$p(0) = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$. If the result is 0, then the state of the qubit after the measurement is $|\psi\rangle \equiv |0\rangle$

$p(1) = \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}$. If the result is 1, then the state of the qubit after the measurement is $|\psi\rangle \equiv |1\rangle$

Some Cautionary Remarks

- The change of state following a measurement is sometimes called the "collapse of the wavefunction", under the common *Copenhagen Intrepretation* of QM
- Immediate remeasurement must give the same answer
- The order of measurements of complicated quantum systems is important
- Measurements in quantum mechanics are represented by Hermitian operators (matrices where $M_{ij} = M_{ji}^*$)
- If, for two Hermitian operators P and Q, $PQ - QP = 0$, then the operators are said to commute, and the order of measurement doesn't matter
- If $PQ - QP \neq 0$, then the operators don't commute and the order does matter.

Conclusions

In this lecture we have:

- defined the state vector and the associated notation
- introduced the inner product and shown how to calculate it in an orthonormal basis
- introduced the idea that quantum measurements are probabilistic
- introduced the idea of wavefunction collapse as a result of measurement

Next lecture we will:

- show how to construct state vectors for multiple qubits
- discuss how quantum states evolve, and what we can do to influence their evolution