Lecture 3: More Quantum Mechanics

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Introduction

Last lecture, we:

- Introduced the notion of a state space and its associated inner product
- Discussed measurement in quantum mechanics

In this lecture we will:

- Discuss how quantum systems evolve, and how we can influence their evolution
- Show how to construct states for multi-qubit systems
- Conclude our discussion of quantum mechanics

The Third Postulate

> The evolution of a closed quantum system is described by a unitary transformation.

- Recall from last time that our quantum system can be described by a state vector, which, for a single qubit, can be written as

$$|\psi\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = a_0 |0\rangle + a_1 |1\rangle$$

- Let's now imagine doing something to this qubit which changes its state to, say

$$|\psi'\rangle = \begin{pmatrix} a_0' \\ a_1' \end{pmatrix} = a_0' |0\rangle + a_1' |1\rangle$$

Transformation Matrices

- We can express the transformation as $U|\psi\rangle = |\psi'\rangle$, where U is some operator
- In our vector-based representation of QM, U can be represented as a matrix
- Recall from the laws of matrix multiplication that a matrix of size $m \times n$ multiplied by a matrix of size $n \times p$ gives a matrix of size $m \times p$
- Since we have $m = n = 2$ and $p = 1$ for a single qubit state, the matrix U must be 2×2:

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_0' \\ a_1' \end{pmatrix}$$

- We then have that $a_0' = u_{11}a_0 + u_{12}a_1$ and $a_1' = u_{21}a_0 + u_{22}a_1$

Example: Quantum NOT Gate

- Recall the definition of the classical NOT gate:

A	Q
0	1
1	0

- Easy to imagine a quantum analogue of this: $\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \mapsto \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$
- Note that in the classical limit, these are equivalent
- We'll represent this using a matrix X
- We have that $x_{11}a_0 + x_{12}a_1 = a_1$ and $x_{21}a_0 + x_{22}a_1 = a_0$
- Must hold for all a_0, a_1
- Only possible if $x_{11} = x_{22} = 0$, $x_{12} = x_{21} = 1$
- So the matrix X representing NOT is $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Unitarity

- Can we use any transformation we like?
- No: only **unitary** transformations are permitted
- A unitary transformation satisfies the condition that $U^\dagger U = I$
- $U^\dagger$ is the *Hermitian conjugate* of U: $U^\dagger_{ij} = U^*_{ji}$
- Is the NOT matrix X unitary? It had better be!

$$X^\dagger = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \text{so}\ X^\dagger X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

- X is unitary and is therefore an allowed transformation
- Note that $U^\dagger U = I$ implies that $U^{-1} \equiv U^\dagger$
- Unitary transformations are therefore always reversible
- So only reversible functions can be implemented

Why Unitarity?

- Why are only unitary transformations allowed?
- The key lies in the way the inner product tranforms
- Consider a state $|\psi'\rangle = U|\psi\rangle$. We must have $\langle\psi|\psi\rangle = 1$, and also $\langle\psi'|\psi'\rangle = 1$ for these to be valid states
- In component notation, $|\psi'\rangle_i = \sum_j u_{ij}|\psi\rangle_j$
- So $\langle\psi'|_i = \sum_j u^*_{ji}\langle\psi|_j$
- In matrix notation, this is $\langle\psi'| = \langle\psi|U^\dagger$ (remember that $\langle\psi'|$ is a row vector, so this is OK)
- We therefore have that

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|\psi\rangle$$

- Unitary transforms are rotations in state space which preserve the length of the state vector (inner product)

The Fourth Postulate

The state space of a composite physical system is the tensor product of the state spaces of the component systems

- There are two parts to this
  - Determining the basis for the composite system
  - Determining the components of the state vector
- Suppose we have two independent state spaces with orthonormal bases $\{|u\rangle\}$ and $\{|v\rangle\}$ respectively
- Bring these together into a single composite system
- The basis states for the composite system are defined as $|w\rangle = |u\rangle \otimes |v\rangle$ for every combination of $|u\rangle$ and $|v\rangle$
- Alternative notations: $|u\rangle|v\rangle$, or $|u, v\rangle$, or simply $|uv\rangle$

A basis for 2-qubit states

- Take two independent qubits
- Each has basis states $\{|0\rangle, |1\rangle\}$
- Then the basis for the composite system is:

$$\begin{aligned} |0\rangle \otimes |0\rangle &\equiv |00\rangle \\ |0\rangle \otimes |1\rangle &\equiv |01\rangle \\ |1\rangle \otimes |0\rangle &\equiv |10\rangle \\ |1\rangle \otimes |1\rangle &\equiv |11\rangle \end{aligned}$$

- The basis contains all the possible combinations of the single qubit states
- Measuring the state of the composite system can therefore yield 00 or 01 or 10 or 11 (as we might expect)
- What are the coefficients of the basis state in the state vector?

Determing the state vector

- Consider two independent states

$$\begin{aligned} |\psi\rangle &= \sum_i \psi_i |i\rangle \\ |\phi\rangle &= \sum_j \phi_j |j\rangle \end{aligned}$$

- The composite state $|\psi\rangle \otimes |\phi\rangle$ is formed as follows:
- First, note that the basis states for the composite system are $\{|ij\rangle\}$
- Then the state of the composite system can be written as

$$|\psi\rangle \otimes |\phi\rangle = \sum_{ij} \psi_i \phi_j |ij\rangle$$

- Again, this is really quite straightforward in practice

Combining two qubits

- Consider two independent qubits in the states

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle; \qquad |\psi_2\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$$

- The basis for the composite system is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$
- The general state is $|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$
- And we simply multiply the coefficients in the single qubit states in the appropriate way:

$$a_{00} = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{5}}; \quad a_{01} = \frac{1}{\sqrt{3}}\frac{2}{\sqrt{5}}; \quad a_{10} = \sqrt{\frac{2}{3}}\frac{1}{\sqrt{5}}; \quad a_{11} = \sqrt{\frac{2}{3}}\frac{2}{\sqrt{5}}$$

- And so $|\psi\rangle = \frac{1}{\sqrt{15}}|00\rangle + \frac{2}{\sqrt{15}}|01\rangle + \sqrt{\frac{2}{15}}|10\rangle + \frac{2\sqrt{2}}{\sqrt{15}}|11\rangle$
- Note that the resulting compound state is still normalised

Entanglement

- Given two independent states, we can always make a composite system from them
- But not all composite systems can be made from independent states!
- Consider the two-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- There are no states $|a\rangle = a_0|0\rangle + a_1|1\rangle$, $|b\rangle = b_0|0\rangle + b_1|1\rangle$ such that $|a\rangle \otimes |b\rangle = |\psi\rangle$
- Why? Since $|\psi\rangle = a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle$
- Then in our state, $a_1b_0 = a_0b_1 = 0$
- But this can only be true if ($a_1 = 0$ or $b_0 = 0$) and ($a_0 = 0$ or $b_1 = 0$)
- This must mean that at least one of a_0b_0 and a_1b_1 must also be zero!

Entanglement 2

- The two qubits in $|\psi\rangle$ are often referred to as being **entangled**
- There is no way of making such a state just by bringing two qubits together
- Entanglement is a pure quantum effect
- We will encounter it again when we discuss quantum teleportation
- Next lecture we will spend some time discussing the Bell states, which are special entangled states with some interesting properties

Conclusions

In this lecture we have:

- Discussed the evolution of quantum states
- Shown how to use unitary operators to describe quantum evolution
- Seen how to construct composite quantum states
- Introduced the idea of entanglement

Next lecture we will:

- Study the properties of qubits in more detail