Lecture 5: Quantum Logic

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Introduction

Last lecture, we:

- discussed various properties of qubit states

In this lecture we will:

- Introduce some common quantum logic gates
- Describe some simple quantum circuits
- Analyse simple circuits

Basic Quantum Logic

- We all know that modern computers are built from networks of logic gates
- Quantum computing uses logic gates in a similar way: networks of them are constructed to perform certain operations
- There are many more possible quantum gates (infinitely many)
- They must all be reversible (due to unitarity)
- So we can represent them with a unitary matrix
- Example: Quantum NOT gate: $\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \mapsto \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$
- The matrix which does this transformation is $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

Quantum Logic Gates

- The quantum analogue of the NOT gate is easy to picture
- The other classical gates that we known and “love” are not so easy to transfer to the quantum world
- All quantum gates must be reversible: classical AND, OR etc are not!
- We will not use these familiar gates as they require effort to be useful, and do not do all that we require
- We will introduce some new gates!

Controlled NOT (CNOT)

- Quantum CNOT gate:

A	B	A'	B'
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

- The transformation is therefore $\begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} \mapsto \begin{pmatrix} a_{00} \\ a_{01} \\ a_{11} \\ a_{10} \end{pmatrix}$
- and is represented by the matrix $N_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Controlled-Controlled NOT

Quantum CCN Gate:

A	B	C	A'	B'	C'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

$\begin{pmatrix} a_{000} \\ a_{001} \\ a_{010} \\ a_{011} \\ a_{100} \\ a_{101} \\ a_{110} \\ a_{111} \end{pmatrix} \mapsto \begin{pmatrix} a_{000} \\ a_{001} \\ a_{010} \\ a_{011} \\ a_{100} \\ a_{101} \\ a_{111} \\ a_{110} \end{pmatrix}$ with matrix $N_{CC} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Controlled Exchange

uantum CEX Gate:

A	B	C	A'	B'	C'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	1	0
1	1	0	1	0	1
1	1	1	1	1	1

$\begin{pmatrix} a_{000} \\ a_{001} \\ a_{010} \\ a_{011} \\ a_{100} \\ a_{101} \\ a_{110} \\ a_{111} \end{pmatrix} \mapsto \begin{pmatrix} a_{000} \\ a_{001} \\ a_{010} \\ a_{011} \\ a_{100} \\ a_{110} \\ a_{101} \\ a_{111} \end{pmatrix}$ with matrix $X_C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Quantum Gates with no Classical Counterpa

- The NOT, CN, CCN and CEX gates are all gates which can exist in bo classical and quantum worlds
- There are many other gates with no classical analogue:
  - The Y and Z gates: $Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  - Together with the X, these are known as the *Pauli matrices*
  - The Hadamard Gate: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
  - H is one of the most useful quantum gates: it takes classical states $(|0\rangle, |1\rangle)$ to to states "halfway" between $|0\rangle$ and $|1\rangle$:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

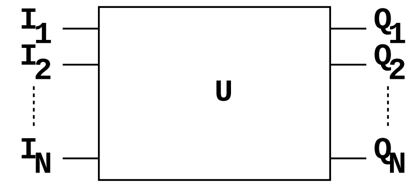
More New Gates

- We also have the Phase Gate: $S = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$
- The T gate: $T = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \end{pmatrix}$
- and the R$_k$ gate: $R_k = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\pi\mathrm{i}/2^k} \end{pmatrix}$
- Note that $S = R_2$ and $T = R_3$
- These seem a bit arbitrary, but all have been found to be useful in various quantum algorithms. We'll be using some of them later.

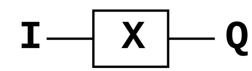
Analysing Quantum Circuits

- So far, we've represented quantum gates with matrices
- It will often be useful to adopt a graphical representation:

$|Q\rangle = U\,|I\rangle \mapsto$



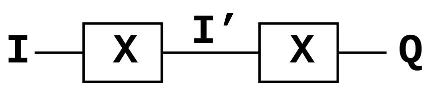
- Each I/O line represents one qubit, e.g, for the X gate:



- The analysis of circuits made from quantum gates is quite straightforward: all we have to do is apply the matrices in the right ord

A Simple Circuit

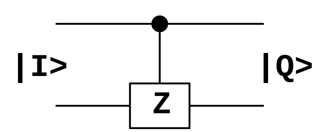
ample: NOT-NOT



- Let $|I\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$. Then $|I'\rangle = X\,|I\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$
- We then have $|Q\rangle = X\,|I'\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$
- This is what we would expect from NOT-NOT: no change
- We write this as $|Q\rangle = X\,|I'\rangle = XX\,|I\rangle = X^2\,|I\rangle$
- Note that $X^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and we can express the entire circuit with a single matrix!

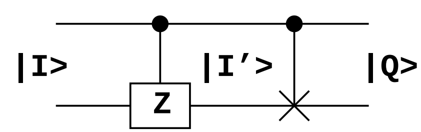
Another Example

In this example, we will use the Controlled-Z gate:



with matrix $Z_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Consider the circuit

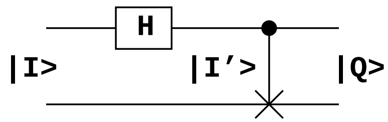


It should be clear that $|I'\rangle = Z_c\,|I\rangle$, and $|Q\rangle = N_c\,|I'\rangle$.
So $|Q\rangle = N_c Z_c\,|I\rangle$ (note the order)

Then $N_c Z_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

A Harder Example

- In the algorithms we will study next week, we will need to be able to analyse circuits which contain both one- and two-input gate. This is a bit more complicated.
- Consider the circuit



- Clearly $|Q\rangle = N_c\,|I'\rangle$, but how do we find matrix M such that $|I'\rangle = M\,|I\rangle$?
- The Hadamard gate H only acts on one bit of our state
- Its action on the first bit is $H\,|0\rangle = \frac{1}{\sqrt{2}}\,(|0\rangle + |1\rangle)$, and $H\,|1\rangle = \frac{1}{\sqrt{2}}\,(|0\rangle - |1\rangle)$
- $|I\rangle$ is defined on the basis $\{|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle\}$

A Harder Example

- The basis states transform in the following way:

$$\begin{aligned}
|00\rangle &\mapsto \frac{1}{\sqrt{2}}\,(|00\rangle + |10\rangle) \\
|01\rangle &\mapsto \frac{1}{\sqrt{2}}\,(|01\rangle + |11\rangle) \\
|10\rangle &\mapsto \frac{1}{\sqrt{2}}\,(|00\rangle - |10\rangle) \\
|11\rangle &\mapsto \frac{1}{\sqrt{2}}\,(|01\rangle - |11\rangle)
\end{aligned}$$

- The matrix for this is $M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

A Harder Example

- So $|I'\rangle = M\,|I\rangle$ and hence $|Q\rangle = M\,|I'\rangle = N_c M\,|I\rangle$. We need to calculate

$$N_c M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

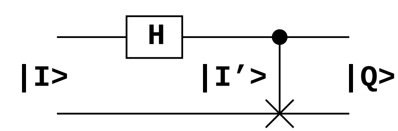
- This is unitary, as required and expected
- This is actually quite an important quantum circuit: it acts on the basis states in the following way:

$$|00\rangle \mapsto \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle) \qquad |01\rangle \mapsto \frac{1}{\sqrt{2}}\,(|01\rangle + |10\rangle)$$

$$|10\rangle \mapsto \frac{1}{\sqrt{2}}\,(|00\rangle - |11\rangle) \qquad |11\rangle \mapsto \frac{1}{\sqrt{2}}\,(|01\rangle - |10\rangle)$$

Creating Bell States

- The new states are all entangled pairs (Bell states): if one of the qubit measured, we automatically know what the other one is without measuring it
- Starting with an initially independent set of qubits, we can form a composite state and pass it through this circuit in order to create entangled pairs



Conclusions

this lecture we have:

- Introduced some common quantum logic gates
- Shown how to analyse networks of quantum logic gates
- Shown how to create entangled pairs using quantum logic gates

xt lecture we will:

- Introduce the Quantum Fourier Transform