

Introduction

Lecture 6: The Quantum Fourier Transform

Dr Iain Styles: I.B.Styles@cs.bham.ac.uk

Last lecture, we:

- showed how to analyse some simple quantum logic circuits

In this lecture we will:

- introduce the discrete Fourier transform (DFT)
- show how the quantum Fourier transform (QFT) emerges from the DFT
- construct a quantum circuit that performs the QFT

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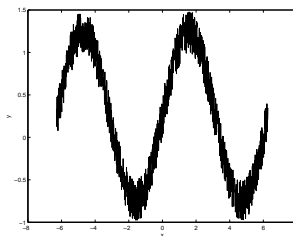
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The Fourier Transform

The Fourier Transform (FT) is one of the most useful mathematical tools in modern science and engineering. It is used, amongst many other things, to:

- remove noise from data
- examine the properties of crystals
- produce holograms

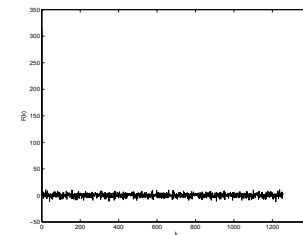
The FT is especially useful when we have something with underlying periodicity: imagine a sine wave with a bit of high-frequency noise



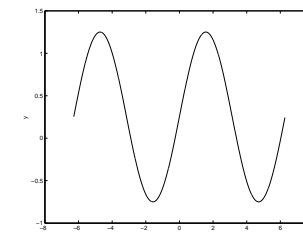
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The Fourier Transform

We can Fourier Transform the data to get a frequency spectrum:



We then remove the high-frequency components (the noise) from the spectrum, and inverse-FT to give a clean set of data:



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The Fourier Transform

- FT allows us to extract the underlying periodic behaviour of a function
- Period finding is the basis for Shor's factoring algorithm, and we will use the QFT in this important application of quantum computing
- We must begin by defining the discrete version of the Fourier Transform, which will form the basis for the quantum algorithm

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An Example Calculation

- Given $x_j = \{1, 2\}$, calculate y_k
- $x_0 = 1, x_1 = 2$, and $N = 2$
- So $y_k = \frac{1}{\sqrt{2}} \sum_{j=0}^1 x_j e^{2\pi i j k / 2}$
- For $k = 0, y_0 = \frac{1}{\sqrt{2}} \sum_{j=0}^1 x_j = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$
- For $k = 1, y_1 = \frac{1}{\sqrt{2}} \sum_{j=0}^1 x_j e^{2\pi i j / 2} = \frac{1}{\sqrt{2}} (1 + 2e^{\pi i}) = -\frac{1}{\sqrt{2}}$
- So it's really not that hard to calculate
- The Fast Fourier Transform (FFT) algorithm of Cooley and Tukey allows us to compute the DFT very rapidly
- The FFT is often used in sound and image processing for the removal of noise
- In general, the FT is useful when there is underlying periodicity
- We will later see that the FT allows us to manipulate quantum state vectors to allow us to measure the result of quantum computations

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Discrete Fourier Transform

- The DFT is a version of the FT which works on discrete data sets
- Mathematically, the DFT is written as

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i j k / N}$$

- Looks formidable, but isn't too hard to calculate
- x_j are complex numbers, with $j = 0 \dots N - 1$
- y_k are complex numbers, with $k = 0 \dots N - 1$
- $i = \sqrt{-1}$; j and k are indices
- An example will show us how to calculate this in practice

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The Quantum Fourier Transform

- Since our state vectors for qubits are just vectors of complex numbers, we should not be surprised to learn that the DFT can be applied to them

- Given a state vector $|\psi\rangle = \sum_{j=0}^{N-1} a_j |j\rangle = \begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \end{pmatrix}$

- We can compute the DFT of this state as

$$F|\psi\rangle = \sum_{k=0}^{N-1} b_k |k\rangle$$

where $b_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} a_j e^{2\pi i j k / N}$

- It can be shown that this is unitary, and so can be implemented

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Example of a QFT

- Consider the 2-qubit state $|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$, which has $N = 4$.
- Then $b_k = \frac{1}{2} \sum_{j=0}^3 a_j e^{2\pi i j k / 4}$, and we have

$$b_0 = \frac{1}{2} \sum_{j=0}^3 a_j = \frac{1}{2} (a_{00} + a_{01} + a_{10} + a_{11})$$

$$b_1 = \frac{1}{2} \sum_{j=0}^3 a_j e^{2\pi i j / 4} = \frac{1}{2} (a_{00} + a_{01} e^{i\pi/2} + a_{10} e^{i\pi} + a_{11} e^{3i\pi/2})$$

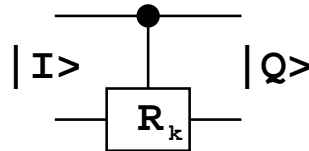
$$b_2 = \frac{1}{2} \sum_{j=0}^3 a_j e^{4\pi i j / 4} = \frac{1}{2} (a_{00} + a_{01} e^{i\pi} + a_{10} e^{2i\pi} + a_{11} e^{3i\pi})$$

$$b_3 = \frac{1}{2} \sum_{j=0}^3 a_j e^{6\pi i j / 4} = \frac{1}{2} (a_{00} + a_{01} e^{3i\pi/2} + a_{10} e^{3i\pi} + a_{11} e^{9i\pi/2})$$

QFT Circuit

- The circuit that implements the QFT is quite simple, but has some new elements in it.
- We need to introduce some new gates:
- The Controlled- R_k gate:

$$R_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i / 2^k} \end{pmatrix}$$



Example of a QFT

- Writing $\omega = e^{\pi i / 2}$, and noting that $\omega^4 = e^{2\pi i} = 1$ and so, e.g. $e^{9i\pi/2} = e^{i\pi/2} = i$, we can write the 2-qubit QFT in matrix form:

$$F = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

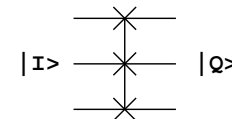
- This can easily be shown to be a unitary operator
- We can build a fairly simple quantum circuit that performs this transformation

QFT Circuit

- We also need a gate which swaps the order of the qubits
- On three qubits, this gate does the following:

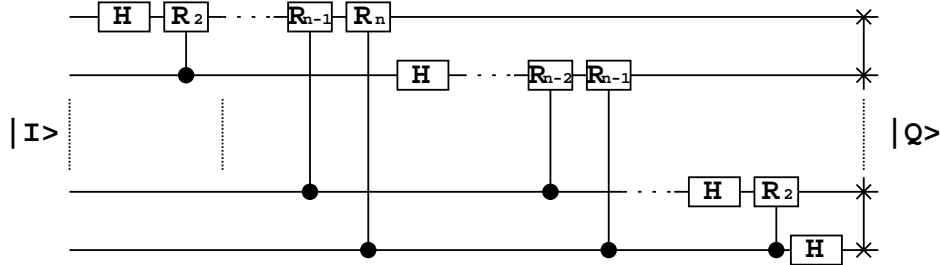
$ 000\rangle \mapsto 000\rangle$	$\mapsto S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$ 001\rangle \mapsto 100\rangle$	
$ 010\rangle \mapsto 010\rangle$	
$ 011\rangle \mapsto 110\rangle$	
$ 100\rangle \mapsto 001\rangle$	
$ 101\rangle \mapsto 101\rangle$	
$ 110\rangle \mapsto 011\rangle$	
$ 111\rangle \mapsto 111\rangle$	

- In the circuit, we will use the symbol:



QFT Circuit

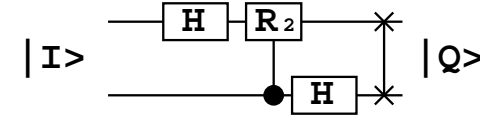
The circuit which performs the QFT is drawn as follows:



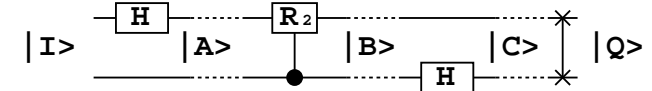
Let's look in detail at an example of this

Two Qubit QFT

The circuit which implements the two-qubit QFT is:



- What is the corresponding transformation matrix?
- We split things up as we have done previously:



- With $|A\rangle = U_1 |I\rangle$, $|B\rangle = U_2 |A\rangle$, $|C\rangle = U_3 |B\rangle$ and $|Q\rangle = U_4 |C\rangle$

- We already know that $U_2 = R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$

Two Qubit QFT

- It is easy to show that

$$U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad U_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad U_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Then we have that $F = U_4 U_3 U_2 U_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$

- This is the same as we obtained by doing the calculation directly, and we have verified that this circuit does indeed perform the QFT

Conclusions

In this lecture we have:

- Introduced the Discrete Fourier Transform
- Shown how to apply it to quantum states (the Quantum Fourier Transform)
- Shown how to implement the QFT using quantum logic gates

Next lecture we will:

- Introduce the first of our applied quantum algorithms, Grover's Algorithm, which can perform searches in time $O(\sqrt{N})$