Quiz 2

1. Remembering that unitary operators obey $U^\dagger U = I$, where $U^\dagger_{ij} = U^*_{ji}$, which of the following are unitary?

$$\text{(a)} \quad \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix} \qquad \text{(b)} \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \qquad \text{(c)} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

2. Given states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \,, \quad |\psi_2\rangle = \frac{1}{\sqrt{3}} |0\rangle + \sqrt{\frac{2}{3}} |1\rangle \,, \quad |\psi_3\rangle = \frac{1}{2} |0\rangle + \frac{\sqrt{3}}{2} |1\rangle$$

Construct the composite states

(a) $|\psi_1\rangle \otimes |\psi_2\rangle$

(b) $|\psi_1\rangle \otimes |\psi_3\rangle$

(c) $|\psi_2\rangle \otimes |\psi_3\rangle$

Solutions

1. Remembering that unitary operators obey $U^\dagger U = I$, where $U^\dagger_{ij} = U^*_{ji}$, which of the following are unitary?

$$\text{(a)} \quad \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix} \quad \text{(b)} \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{(c)} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(a) $U = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}$. Therefore $U^\dagger = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}$ and

$U^\dagger U = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix} \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. NOT UNITARY.

(b) $U = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. Therefore $U^\dagger = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ and

$U^\dagger U = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. NOT UNITARY.

(c) $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Therefore $U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and

$U^\dagger U = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. UNITARY.

$$
\begin{aligned}
2(\mathrm{a}) \quad |\psi_1\rangle \otimes |\psi_2\rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}}|01\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}|10\rangle + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}}|11\rangle \\
&= \frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{3}}|11\rangle \\
2(\mathrm{b}) \quad |\psi_1\rangle \otimes |\psi_3\rangle &= \frac{1}{\sqrt{2}}\frac{1}{2}|00\rangle + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}|01\rangle + \frac{1}{\sqrt{2}}\frac{1}{2}|10\rangle + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}|11\rangle \\
&= \frac{1}{2\sqrt{2}}|00\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle
\end{aligned}
$$

$$
2(c) \quad |\psi_2\rangle \otimes |\psi_3\rangle = \frac{1}{\sqrt{3}}\frac{1}{2}|00\rangle + \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}|01\rangle + \sqrt{\frac{2}{3}}\frac{1}{2}|10\rangle + \sqrt{\frac{2}{3}}\frac{\sqrt{3}}{2}|11\rangle
$$

$$
= \frac{1}{2\sqrt{3}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{2}}{2\sqrt{3}}|10\rangle + \frac{\sqrt{2}}{2}|11\rangle
$$