Introduction to Molecular and Quantum Computations: Exercise Sheet 1

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January 13, 2009

1. Properties of State Vectors

a) Express the following qubit states in vector form:

i) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
ii) $|\psi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$
iii) $|\psi\rangle = \frac{1}{3}|0\rangle + \frac{2\sqrt{2}}{3}|1\rangle$

b) Express the following qubit states in sum form, $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$:

i) $\frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} \\ -1 \end{pmatrix}$
ii) $\frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -\sqrt{2} \end{pmatrix}$
iii) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

c) Calculate the bra state $\langle\psi|$ associated with the states $|\psi\rangle$ below. Express your answer in both vector and sum forms.

i) $|\psi\rangle = \frac{2}{\sqrt{7}}|0\rangle + \mathrm{i}\sqrt{\frac{3}{7}}|1\rangle$
ii) $|\psi\rangle = \frac{1+\mathrm{i}}{\sqrt{3}}|0\rangle + \frac{\mathrm{i}}{\sqrt{3}}|1\rangle$
iii) $-\mathrm{i}|0\rangle$

d) Three single-qubit states are defined as $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, $|\psi_3\rangle = \frac{3\mathrm{i}}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle$. Calculate:

i) $\langle\psi_1|\psi_1\rangle$
ii) $\langle\psi_2|\psi_2\rangle$
iii) $\langle\psi_3|\psi_3\rangle$
iv) $\langle\psi_1|\psi_2\rangle$
v) $\langle\psi_1|\psi_3\rangle$
vi) $\langle\psi_2|\psi_3\rangle$

Which one is not an allowed qubit state? Which pair are orthogonal?

2. Measurement of Qubits

a) What are the probabilities that a measurement will yield 0 or 1 for the following states?

i) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
ii) $|\psi\rangle = \frac{1}{\sqrt{6}}|0\rangle + \mathrm{i}\sqrt{\frac{5}{6}}|1\rangle$
iii) $|\psi\rangle = \frac{1-\mathrm{i}}{\sqrt{3}}|0\rangle + \frac{\mathrm{i}}{\sqrt{3}}|1\rangle$

b) For the following states, what is the probability that the post-measurement state is $|0\rangle$?

i) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
ii) $|\psi\rangle = \frac{2-\mathrm{i}}{\sqrt{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle$
iii) $|\psi\rangle = |1\rangle$

c) Measurements of the following qubit states yield the results R shown below. What is the probability that an immediate subsequent measurement will be 0? What is the probability that an immediate subsequent measurement will be 1?

i) $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle - \mathrm{i}\sqrt{\frac{2}{3}}|1\rangle$ Measurement $\mapsto 1$
ii) $|\psi\rangle = \frac{1}{\sqrt{7}}|0\rangle + \sqrt{\frac{6}{7}}|1\rangle$ Measurement $\mapsto 0$
iii) $|\psi\rangle = |1\rangle$ Measurement $\mapsto 1$

3. Evolution of Qubit States

a) Which of the following are allowable evolution operators?

i) $M = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$
ii) $M = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}$
iii) $M = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}$
iv) $M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$

4. Composite States

a) For $|\psi_1\rangle = |0\rangle$, $|\psi_2\rangle = |1\rangle$, $|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, calculate:

i) $|\psi_1\rangle \otimes |\psi_2\rangle$
ii) $|\psi_1\rangle \otimes |\psi_3\rangle$
iii) $|\psi_2\rangle \otimes |\psi_3\rangle$
iv) $|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle$

b) Which of the following states cannot be constructed from two independent qubits? If they can, write down the two independent states that can form the given composite state.

i) $|\psi\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$
ii) $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$
iii) $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$
iv) $|\psi\rangle = \frac{1}{2}(|010\rangle - |011\rangle - |001\rangle + |000\rangle)$

5. A more involved question

A qubit is initially programmed to state $|\psi_1\rangle = |0\rangle$. A unitary transformation $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is applied to the qubit. The resulting state is combined into a new composite state with a second qubit $|\psi_2\rangle = \frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$. The two qubits are then measured. What is the most likely result of the measurement?