Solutions to Exercise Sheet 1

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1. Properties of State Vectors

a) Express the following qubit states in vector form:

i) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
ii) $|\psi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$
iii) $|\psi\rangle = \frac{1}{3}|0\rangle + \frac{2\sqrt{2}}{3}|1\rangle$

Solutions

i) $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ ii) $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$ iii) $\frac{1}{3}\begin{pmatrix}1\\2\sqrt{2}\end{pmatrix}$

b) Express the following qubit states in sum form, $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$:

i) $\frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{i}\\-1\end{pmatrix}$
ii) $\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-\sqrt{2}\end{pmatrix}$
iii) $\begin{pmatrix}0\\1\end{pmatrix}$

Solutions

i) $\frac{1}{\sqrt{2}}(\mathrm{i}|0\rangle - |1\rangle)$
ii) $-\frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$
iii) $|1\rangle$

c) Calculate the bra state $\langle\psi|$ associated with the states $|\psi\rangle$ below. Express your answer in both vector and sum forms.

i) $|\psi\rangle = \frac{2}{\sqrt{7}}|0\rangle + \mathrm{i}\sqrt{\frac{3}{7}}|1\rangle$
ii) $|\psi\rangle = \frac{1+\mathrm{i}}{\sqrt{3}}|0\rangle + \frac{\mathrm{i}}{\sqrt{3}}|1\rangle$
iii) $|\psi\rangle = -\mathrm{i}|0\rangle$

Solutions

i) $\langle\psi| = \frac{2}{\sqrt{7}}\langle 0| - \mathrm{i}\sqrt{\frac{3}{7}}\langle 1| = \begin{pmatrix}\frac{2}{\sqrt{7}} & -\mathrm{i}\sqrt{\frac{3}{7}}\end{pmatrix}$
ii) $\langle\psi| = \frac{1-\mathrm{i}}{\sqrt{3}}\langle 0| - \frac{\mathrm{i}}{\sqrt{3}}\langle 1| = \begin{pmatrix}\frac{1-\mathrm{i}}{\sqrt{3}} & -\frac{\mathrm{i}}{\sqrt{3}}\end{pmatrix}$
iii) $\langle\psi| = \mathrm{i}\langle 0| = \begin{pmatrix}\mathrm{i} & 0\end{pmatrix}$

d) Three single-qubit states are defined as $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, $|\psi_3\rangle = \frac{3\mathrm{i}}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle$. Calculate:

i) $\langle\psi_1|\psi_1\rangle$
ii) $\langle\psi_2|\psi_2\rangle$
iii) $\langle\psi_3|\psi_3\rangle$
iv) $\langle\psi_1|\psi_2\rangle$
v) $\langle\psi_1|\psi_3\rangle$
vi) $\langle\psi_2|\psi_3\rangle$

Which one is not an allowed qubit state? Which pair are orthogonal?

Solutions

i) $\langle\psi_1|\psi_1\rangle = 1$
ii) $\langle\psi_2|\psi_2\rangle = 1$
iii) $\langle\psi_3|\psi_3\rangle = 2$
iv) $\langle\psi_1|\psi_2\rangle = 0$
v) $\langle\psi_1|\psi_3\rangle = \frac{1}{\sqrt{10}}(1 + 3\mathrm{i})$
vi) $\langle\psi_2|\psi_3\rangle = \frac{1}{\sqrt{10}}(-1 + 3\mathrm{i})$

$|\psi_3\rangle$ is an illegal state as it is not normalised. $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal. Note that they could be used as an alternative basis for a single qubit state.

2. Measurement of Qubits

a) What are the probabilities that a measurement will yield 0 or 1 for the following states?

i) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
ii) $|\psi\rangle = \frac{1}{\sqrt{6}}|0\rangle + \mathrm{i}\sqrt{\frac{5}{6}}|1\rangle$
iii) $|\psi\rangle = \frac{1-\mathrm{i}}{\sqrt{3}}|0\rangle + \frac{\mathrm{i}}{\sqrt{3}}|1\rangle$

Solutions

i) $p(0) = \frac{1}{2}$, $p(1) = \frac{1}{2}$ ii) $p(0) = \frac{1}{6}$, $p(1) = \frac{5}{6}$ iii) $p(0) = \frac{2}{3}$, $p(1) = \frac{1}{3}$

b) For the following states, what is the probability that the post-measurement state is $|0\rangle$?

i) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
ii) $|\psi\rangle = \frac{2-\mathrm{i}}{\sqrt{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle$
iii) $|\psi\rangle = |1\rangle$

Solutions

i) $p(0) = \frac{1}{2}$ ii) $p(0) = \frac{5}{7}$ iii) $p(0) = 0$

c) Measurements of the following qubit states yield the results R shown below. What is the probability that an immediate subsequent measurement will be 0? What is the probability that an immediate subsequent measurement will be 1?

i) $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle - \mathrm{i}\sqrt{\frac{2}{3}}|1\rangle$ Measurement $\mapsto 1$

ii) $|\psi\rangle = \frac{1}{\sqrt{7}}|0\rangle + \sqrt{\frac{6}{7}}|1\rangle$ Measurement $\mapsto 0$

iii) $|\psi\rangle = |1\rangle$ Measurement $\mapsto 1$

Solutions

i) $p(0) = 0, p(1) = 1$ ii) $p(0) = 1, p(1) = 0$ iii) $p(0) = 0, p(1) = 1$

3. Evolution of Qubit States

a) Which of the following are allowable evolution operators?

i) $M = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$

ii) $M = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}$

iii) $M = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}$

iv) $M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$

Solutions

i) $M = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \mapsto M^\dagger = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \mapsto M^\dagger M = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

ii) $M = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix} \mapsto M^\dagger = \begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix} \mapsto M^\dagger M = \begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 4 \end{pmatrix}$

iii) $M = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix} \mapsto M^\dagger = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix} \mapsto M^\dagger M = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

iv) $M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \mapsto M^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \mapsto M^\dagger M = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So matrices (i) and (iv) are unitary, since $M^\dagger M = I$.

4. Composite States

i) $|\psi_1\rangle \otimes |\psi_2\rangle$

ii) $|\psi_1\rangle \otimes |\psi_3\rangle$

iii) $|\psi_2\rangle \otimes |\psi_3\rangle$

iv) $|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle$

Solutions

i) $|\psi_1\rangle \otimes |\psi_2\rangle = |01\rangle$

ii) $|\psi_1\rangle \otimes |\psi_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)$

iii) $|\psi_2\rangle \otimes |\psi_3\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$

iv) $|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle = \frac{1}{\sqrt{2}}(|010\rangle - |011\rangle)$

b) Which of the following states cannot be constructed from two independent qubits? If they can, write down the two independent states that can form the given composite state.

i) $|\psi\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$
ii) $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$
iii) $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$
iv) $|\psi\rangle = \frac{1}{2}(|010\rangle - |011\rangle - |001\rangle + |000\rangle)$

Solutions

i) $|\psi\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) = \left[\frac{1}{\sqrt{2}}(-|0\rangle - |1\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)\right]$
ii) $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is an entangled state and cannot be decomposed.
iii) $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]$
iv) $|\psi\rangle = \frac{1}{2}(|010\rangle - |011\rangle - |001\rangle + |000\rangle) = |0\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]$

5. A more involved question

A qubit is initially programmed to state $|\psi_1\rangle = |0\rangle$. A unitary transformation $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is applied to the qubit. The resulting state is combined into a new composite state with a second qubit $|\psi_2\rangle = \frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$. The two qubits are then measured. What is the most likely result of the measurement?

Solution

The initial state of the qubit is $|\psi_1\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We apply transformation H to this state, giving $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. We now combine this result with state $|\psi_2\rangle = \frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$ as follows: $\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \otimes \left[\frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle\right] = \frac{1}{\sqrt{6}}|00\rangle - \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle$. We then have that $p(00) = \frac{1}{6}$, $p(01) = \frac{1}{3}$, $p(10) = \frac{1}{6}$, $p(11) = \frac{1}{3}$. So the most likely result of the measurement is either 01 or 11.