Solutions to Exercise Sheet 2

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1. Measurement of multi-qubit states

a) A system containing two qubits is prepared in the following states:

i) $|\psi\rangle = \frac{1}{3}|00\rangle + \frac{2}{3}|10\rangle - \frac{2}{3}|11\rangle$
ii) $|\psi\rangle = \frac{\mathrm{i}}{2}|00\rangle - \frac{1}{\sqrt{3}}|10\rangle + \frac{1}{2}|01\rangle + \frac{\mathrm{i}}{\sqrt{6}}|11\rangle$
iii) $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

The first qubit is measured. What is the probability that the result is 0? What is the probability that the result is 1? For each possible result, write down the post-measurement state, and calculate the probability that a measurement of the second qubit will give 0 and 1. Write down the states after the second measurement.

Solutions

i) $|\psi\rangle = \frac{1}{3}|00\rangle + \frac{2}{3}|10\rangle - |2\rangle 3 |11\rangle$

- The first qubit is measured giving result 0 with $p(0) = \frac{1}{9}$ and post-measurement state $|\psi'\rangle = |00\rangle$. A subsequent measurement of the second bit gives $p(0) = 1$ and $p(1) = 0$ with post-measurement state $|00\rangle$. Note that since $p(1) = 0$, the post-measurement state $|01\rangle$ is not possible.
- The first qubit is measured giving result 1 with $p(1) = \frac{8}{9}$ and post-measurement state $|\psi'\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$. A subsequent measurement of the second bit gives $p(0) = p(1) = \frac{1}{2}$ with post-measurement states $|10\rangle$ and $|11\rangle$ respectively.

ii) $|\psi\rangle = \frac{\mathrm{i}}{2}|00\rangle - \frac{1}{\sqrt{3}}|10\rangle + \frac{1}{2}|01\rangle + \frac{\mathrm{i}}{\sqrt{6}}|11\rangle$

- The first qubit is measured giving result 0 with $p(0) = \frac{1}{2}$ and post-measurement state $|\psi'\rangle = \frac{\mathrm{i}}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$. A subsequent measurement of the second bit gives $p(0) = p(1) = \frac{1}{2}$ with post-measurement states $|00\rangle$ and $|01\rangle$ respectively.
- The first qubit is measured giving result 1 with $p(1) = \frac{1}{2}$ and post-measurement state $|\psi'\rangle = -\sqrt{\frac{2}{3}}|10\rangle + \mathrm{i}\frac{1}{\sqrt{3}}|11\rangle$. A subsequent measurement of the second bit gives $p(0) = \frac{2}{3}$ with post-measurement state $|10\rangle$ and $p(1) = \frac{1}{3}$ with post-measurement state $|11\rangle$.

iii) $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

- The first qubit is measured giving result 0 with $p(0) = \frac{1}{2}$ and post-measurement state $|\psi'\rangle = |00\rangle$. A subsequent measurement of the second bit gives $p(0) = 1$ with post-measurement state $|00\rangle$ and $p(1) = 0$ (no post-measurement state as zero probability of this result).
- The first qubit is measured giving result 1 with $p(1) = \frac{1}{2}$ and post-measurement state $|\psi'\rangle = |11\rangle$. A subsequent measurement of the second bit gives $p(1) = 1$ with post-measurement state $|11\rangle$ and $p(0) = 0$ (no post-measurement state as zero probability of this result).

Note that this is an entangled state, and that upon measuring the first qubit, we fix the value of the second qubit without measuring it.

2. The No-Cloning Theorem

a) Imagine we can define a unitary operator U that can copy the qubit basis states $|0\rangle$ and $|1\rangle$:

$$U|0\rangle|0\rangle = |0\rangle|0\rangle \qquad U|1\rangle|0\rangle = |1\rangle|1\rangle$$

Can U be used to copy $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$? Verify using an explicit calculation.

b) Imagine we can define a unitary operator U that can copy the qubit states $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$:

$$U|\psi_1\rangle|0\rangle = |\psi_1\rangle|\psi_1\rangle \qquad U|\psi_2\rangle|0\rangle = |\psi_2\rangle|\psi_2\rangle$$

Can U be used to copy $|0\rangle$ and $|1\rangle$? Verify using an explicit calculation.

Solutions:

a) U will not copy $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. We can prove this easily.

$$\begin{aligned} U|\psi\rangle|0\rangle &= U\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}|0\rangle\right) \\ &= \frac{1}{\sqrt{2}}(U|0\rangle|0\rangle + U|1\rangle|0\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \end{aligned}$$

But $|\psi\rangle|\psi\rangle = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) = \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)$, and so $U|\psi\rangle|0\rangle \neq |\psi\rangle|\psi\rangle$, and the no-cloning theorem holds.

b) U will not copy $|0\rangle$ and $|1\rangle$. Again, the proof is easy. First, we note that $ket0 = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ and $ket1 = \frac{1}{\sqrt{2}}(|\psi_1\rangle - |\psi_2\rangle)$. Then

$$\begin{aligned} U|0\rangle|0\rangle &= U\left(\frac{|\psi_1\rangle + |\psi_2\rangle}{\sqrt{2}}|0\rangle\right) \\ &= \frac{1}{\sqrt{2}}(U|\psi_1\rangle|0\rangle + U|\psi_2\rangle|0\rangle) \\ &= \frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_1\rangle + |\psi_2\rangle|\psi_2\rangle) \end{aligned}$$

But $|0\rangle|0\rangle = \frac{1}{2}(|\psi_1\rangle + |\psi_2\rangle)(|\psi_1\rangle + |\psi_2\rangle) = \frac{1}{2}(|\psi_1\rangle|\psi_1\rangle + |\psi_1\rangle|\psi_2\rangle + |\psi_2\rangle|\psi_1\rangle + |\psi_2\rangle|\psi_2\rangle)$, and so $U|0\rangle|0\rangle \neq |0\rangle|0\rangle$, and the no-cloning theorem holds.

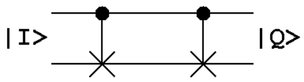
Similarly,

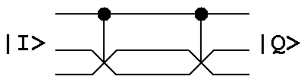
$$\begin{aligned} U|1\rangle|0\rangle &= U\left(\frac{|\psi_1\rangle - |\psi_2\rangle}{\sqrt{2}}|0\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(U|\psi_1\rangle|0\rangle - U|\psi_2\rangle|0\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(|\psi_1\rangle|\psi_1\rangle - |\psi_2\rangle|\psi_2\rangle\right) \end{aligned}$$

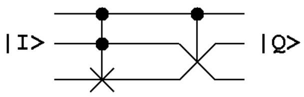
But $|1\rangle|1\rangle = \frac{1}{2}(|\psi_1\rangle - |\psi_2\rangle)(|\psi_1\rangle - |\psi_2\rangle) = \frac{1}{2}(|\psi_1\rangle|\psi_1\rangle - |\psi_1\rangle|\psi_2\rangle - |\psi_2\rangle|\psi_1\rangle + |\psi_2\rangle|\psi_2\rangle)$, and so $U|1\rangle|0\rangle \neq |1\rangle|1\rangle$, and the no-cloning theorem holds.

3. Practice with Quantum Logic.

For each of the following quantum circuits, find the matrix M describing the evolution of the system $|Q\rangle = M|I\rangle$. For each case, check that M is unitary.

a) 

b) 

c) 

Solutions

a) Clearly $M = N_c N_c$, where $N_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Therefore:

$$M = N_c N_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the identity matrix and is obviously unitary.

b) Clearly $M = X_c X_c$, where $X_c = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$. Therefore

$$M = X_c X_c = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

This is the identity matrix and is obviously unitary.

c) Clearly $M = X_c N_{cc}$, where X_c is defined in the previous question and $N_{cc} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{pmatrix}$

Therefore

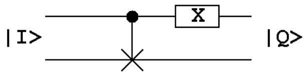
$$M = X_c N_{cc} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0 \end{pmatrix}$$

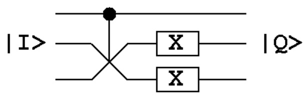
This is not obviously unitary, so we must check it:

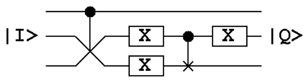
$$M^\dagger M = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

So M is unitary.

4. More Quantum Logic For each of the following quantum circuits, find the matrix M describing the evolution of the system $|Q\rangle = M\,|I\rangle$. For each case, check that M is unitary.

a) 

b) 

c) 

Solutions

a) We have to split this into two parts: First, the CN gate, which has matrix N_c as defined previously. Then the first qubit passes through X and the second is unchanged, which we will describe by some matrix U. This has the following effect:

$$\begin{aligned} |00\rangle &\mapsto |10\rangle \\ |01\rangle &\mapsto |11\rangle \\ |10\rangle &\mapsto |00\rangle \\ |11\rangle &\mapsto |01\rangle \end{aligned} \qquad \text{leading to } U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We then have that

$$M = UN_c = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It is easy to check that this is, indeed, unitary.

b) We have to split this into two parts: First, the CEX gate, with matrix X_c as defined previously. Then the first qubit continues unchanged, which the other two pass through X gates. This is described by a matrix U which has the following effect:

$$\begin{aligned} |000\rangle &\mapsto |011\rangle \\ |001\rangle &\mapsto |010\rangle \\ |010\rangle &\mapsto |001\rangle \\ |011\rangle &\mapsto |000\rangle \\ |100\rangle &\mapsto |111\rangle \\ |101\rangle &\mapsto |110\rangle \\ |110\rangle &\mapsto |101\rangle \\ |111\rangle &\mapsto |100\rangle \end{aligned} \qquad \text{leading to } U = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We then have that

$$M = UX_c = \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} = \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix}$$

This is easily verified as unitary.

c) We have already done most of this question. The first part of the evolution is done by the solution to the previous question, which we will call U_1. The second part is related to part a) of this question, and it is easy to see that it has the following effect:

$$\begin{aligned} |000\rangle &\mapsto |010\rangle \\ |001\rangle &\mapsto |011\rangle \\ |010\rangle &\mapsto |001\rangle \\ |011\rangle &\mapsto |000\rangle \\ |100\rangle &\mapsto |110\rangle \\ |101\rangle &\mapsto |111\rangle \\ |110\rangle &\mapsto |101\rangle \\ |111\rangle &\mapsto |100\rangle \end{aligned} \qquad \text{leading to } U_2 = \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}$$

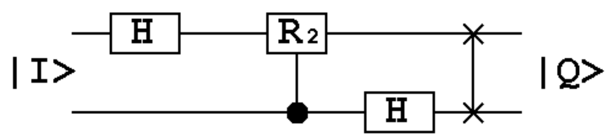
Therefore,

$$M = U_2U_1 = \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0 \end{pmatrix}$$

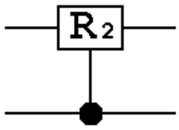
Which is unitary.

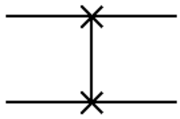
5. An Important Circuit

Find the matrix describing the following circuit:



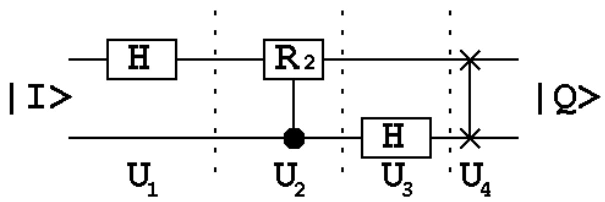
You will need the following matrices for the individual components:





$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution We split the circuit into four parts as follows:



It is then easy to find the matrices:

$$U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$U_2 = R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{pmatrix}$$

$$U_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$U_4 = S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the full matrix is

$$M = U_4 U_3 U_2 U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{pmatrix}$$

This circuit is used to implement the 2-qubit quantum Fourier transform, which we will study in lecture 7.