Solutions to Exercise Sheet 3

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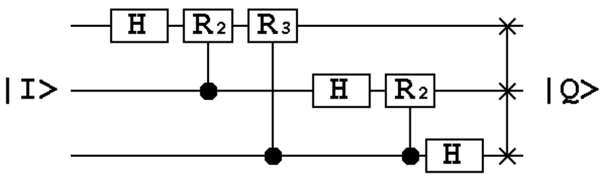
1. Three Qubit Quantum Fourier Transform

The circuit that implements the Quantum Fourier Transform on three qubits is



which is represented by the matrix

$$F_3 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix},$$

where $\omega = \mathrm{e}^{\pi \mathrm{i}/4}$. Show that this matrix correctly describes the three-qubit QFT by explicitly computing the QFT of $|\psi\rangle = \sum_{j=0}^{7} x_j |j\rangle$.

Solution

Recall that the QFT is defined as

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \mathrm{e}^{2\pi \mathrm{i} jk/N}$$

Then for our specific case,

$$y_k = \frac{1}{2\sqrt{2}} \sum_{j=0}^{7} x_j \omega^{jk}$$

So we have

$$
\begin{aligned}
y_0 &= \frac{1}{2\sqrt{2}}\sum_{j=0}^{7} x_j = \frac{1}{2\sqrt{2}}\left(a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7\right) \\
y_1 &= \frac{1}{2\sqrt{2}}\sum_{j=0}^{7} x_j\omega^j = \frac{1}{2\sqrt{2}}\left(a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + a_5\omega^5 + a_6\omega^6 + a_7\omega^7\right) \\
y_2 &= \frac{1}{2\sqrt{2}}\sum_{j=0}^{7} x_j\omega^{2j} = \frac{1}{2\sqrt{2}}\left(a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4 + a_5\omega^2 + a_6\omega^4 + a_7\omega^6\right) \\
y_3 &= \frac{1}{2\sqrt{2}}\sum_{j=0}^{7} x_j\omega^{3j} = \frac{1}{2\sqrt{2}}\left(a_0 + a_1\omega^3 + a_2\omega^6 + a_3\omega + a_4\omega^4 + a_5\omega^7 + a_6\omega^2 + a_7\omega^5\right) \\
y_4 &= \frac{1}{2\sqrt{2}}\sum_{j=0}^{7} x_j\omega^{4j} = \frac{1}{2\sqrt{2}}\left(a_0 + a_1\omega^4 + a_2 + a_3\omega^4 + a_4 + a_5\omega^4 + a_6 + a_7\omega^4\right) \\
y_5 &= \frac{1}{2\sqrt{2}}\sum_{j=0}^{7} x_j\omega^{5j} = \frac{1}{2\sqrt{2}}\left(a_0 + a_1\omega^5 + a_2\omega^2 + a_3\omega^7 + a_4\omega^4 + a_5\omega + a_6\omega^6 + a_7\omega^3\right) \\
y_6 &= \frac{1}{2\sqrt{2}}\sum_{j=0}^{7} x_j\omega^{6j} = \frac{1}{2\sqrt{2}}\left(a_0 + a_1\omega^6 + a_2\omega^4 + a_3\omega^2 + a_4 + a_5\omega^6 + a_6\omega^4 + a_7\omega^2\right) \\
y_7 &= \frac{1}{2\sqrt{2}}\sum_{j=0}^{7} x_j\omega^{7j} = \frac{1}{2\sqrt{2}}\left(a_0 + a_1\omega^7 + a_2\omega^6 + a_3\omega^5 + a_4\omega^4 + a_5\omega^3 + a_6\omega^2 + a_7\omega\right)
\end{aligned}
$$

, where I have noted that $\omega^8 = \mathrm{e}^{2\pi\mathrm{i}} = 1$. This is the same result as we get from multiplying the state vector for $|\psi\rangle$ by the matric F_3, and so we have shown their equivalence.

2. The Inverse Quantum Fourier Transform

Given the matrix for the two-qubit QFT,

$$
F_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{pmatrix},
$$

deduce the matrix that performs the *inverse* QFT, F_2^{-1}. Can you deduce the general sum form of the inverse QFT from this matrix?

Solution

We use the fact that F_2 must be a unitary operator to observe that $F_2^\dagger F_2 = I$. Then it must be true that $F_2^{-1} = F_2^\dagger$. Therefore we can simply write down the matrix for F_2^{-1}:

$$
F_2^{-1} = \frac{1}{\sqrt{2}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{pmatrix}.
$$

Now, since the sum form of the QFT is given by $y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \mathrm{e}^{2\pi \mathrm{i} jk/N}$, we note that the matrix elements of F_2 are simply $(F_2)_{jk} = \frac{1}{\sqrt{N}} e^{2\pi \mathrm{i} jk/N}$. By comparison, we observe that the matrix elements of the inverse QFT are $(F_2^{-1})_{jk} = \frac{1}{\sqrt{N}} e^{-2\pi \mathrm{i} jk/N}$, and so the inverse QFT is

$$x_j = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} y_k \mathrm{e}^{-2\pi \mathrm{i} jk/N}.$$