Introduction to Natural Computation

Lecture 16

Small Worlds

Alberto Moraglio
It’s a small world!
Overview of the Lecture

- The Small World Phenomenon
- Explaining the Small World Phenomenon
- Small World Networks
- Building Small World Networks
In the 1960’s Travers and Milgram performed an experiment…
Small World Phenomenon

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- 196 people in Nebraska and Boston were randomly selected.
- Each *starter person* in Nebraska was given a document, which named arbitrary *target person* in Boston, and asked the recipient of the document to forward it on.
- Each person was only allowed to forward the document to someone with whom they were on first name terms.
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In the world at large, this is expected to be around 6, hence the phrase *six degrees of separation*. 
The Kevin Bacon Game

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How to play...

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- Think of an actor or actress.
- If they have *ever* been in a film with Kevin Bacon, then they have a *Bacon Number* of one.
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- Elvis Presley $\Rightarrow$ 2.
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Some Bacon Numbers

- Elvis Presley → 2.
- Daniel Radcliffe → 2.
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It is claimed that no one who has been in an American film, ever has a Bacon Number of greater than four!
An exhaustive survey of the Internet Movie Database (imdb.com) determined that the highest finite Bacon Number (for any nationality) is eight.

But due to the skew, the mean is 2.981.

Out of over 1.2 million people!

Calculate Bacon numbers at http://oracleofbacon.org/, or for even more fun, take a look at http://en.wikipedia.org/wiki/Erdos-Bacon_number
Explaining the Small World Phenomenon in Social Networks

A social network is a network (i.e. a graph), where nodes represent people (or animals, or computers...) and edges represent social acquaintance, friendship etc.

A social network exhibits the small-world phenomenon if, roughly speaking, any two individuals in the network are likely to be connected through a short sequence of intermediate acquaintances.
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...and they also cluster.
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The degree of nodes appeared to follow a power-law distribution!

The vast majority of nodes have a small number of connections, but a small number have many.
Examples of Small World Networks

“The small world phenomenon is pervasive in networks arising in nature and technology, and a fundamental ingredient in the structural evolution of the World Wide Web” – Kleinberg [2].

Identified examples include:

- The power grid of the western United States,
- The neural network of the worm *Caenorhabditis elegans*,
- The collaboration graph of film actors.
The Internet is a Small World Network
Consider a Regular Lattice

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Rewiring the Lattice

- We can rewire each edge at random with probability $p$.
- This allows us to tune the graph between regularity ($p = 0$) and randomness ($p = 1$).
- We can then investigate the intermediate region $0 < p < 1$. 
Building Small World Networks: Method 1

Rewiring Method

1. Begin with a ring of $n$ vertices, each connected to its $k$ nearest neighbours by undirected edges.
2. Choose a vertex at random.
3. Select the edge that connects the vertex to its nearest neighbour in a clockwise sense.
4. With probability $p$, reconnect this edge to a vertex chosen uniformly at random over the entire ring, with duplicate edges forbidden; otherwise we leave the edge in place.
5. Move clockwise one vertex around the ring and repeat steps 3 and 4, until one lap is completed.

(The current vertex is now the one at which we started.)

6. Select the edge that connects the vertex to its second nearest neighbour in a clockwise sense.
7. With probability $p$, reconnect this edge to a vertex chosen uniformly at random over the entire ring, with duplicate edges forbidden; otherwise we leave the edge in place.
8. Move clockwise one vertex around the ring and repeat steps 6 and 7, until one lap is completed.

Repeat the process, circulating the ring and proceeding outward to more distant neighbours after each lap, until every edge in the original lattice has been considered once.

Since there are $nk/2$ edges in the entire graph, the rewiring process takes $k/2$ laps.
Building Small World Networks: Method 1

Example for $n = 20$ and $k = 4$:

Describing Small World Networks

The structural properties of these networks may be quantified by their characteristic path length, and clustering coefficient.
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### Characteristic path length

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It is defined as the number of edges in the shortest path between two vertices, averaged over all pairs of vertices.
Describing Small World Networks

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Describing Small World Networks

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### Characteristic path length

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It is defined as follows. Suppose that a vertex $v$ has $k_v$ neighbours; then at most $k_v(k_v + 1)/2$ edges can exist between them (this occurs when every neighbour of $v$ is connected to every other neighbour of $v$). Let $C_v$ denote the fraction of these allowable edges that actually exist. Define $C$ as the average of $C_v$ over all $v$.

It may help to think about these statistics in terms of friendship networks:

- $L$ is the average number of friendships in the shortest chain connecting two people;
- $C_v$ reflects the extent to which friends of $v$ are also friends of each other, and thus $C$ measures the cliquishness of a typical friendship circle.
Average data over 20 random realizations of the rewiring process. All the graphs have $n = 1000$ vertices and a degree of $k = 10$ edges per vertex.
Rewiring a Regular Lattice

Observe:
- a rapid drop in $L(p)$ as $p$ grows,
- that $C(p)$ remains almost constant at its value for the regular lattice during this drop.

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Small world networks are **highly clustered**, like regular lattices, yet have **small characteristic path lengths**, like random graphs.

These experiments reveal that there is a broad range for $p$ over which:

- $L(p)$ is almost as small as $L(1)$ (a random network),
- But $C(p) \gg C(1)$.

These **small world networks** result from the immediate drop in $L(p)$ caused by the introduction of a few long range edges, or *short cuts*.
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- This highlights the importance of the **average number of hops** to get from one node in the network to another.
- This in turn relates to the time taken to send a message across the network.
- Even a small number of short cuts can significantly reduce the average number of hops.
- A fully random network does not improve this much further.
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Recall that the Internet is structured similarly to this... makes sense!
Other Real World Examples and Impact

- Enhanced signal-propagation speed,
- Easier to synchronise (e.g. think about firefly neighbourhoods),
- Infectious diseases spread more easily in small-world networks than in regular lattices.

Table 1 Empirical examples of small-world networks

<table>
<thead>
<tr>
<th></th>
<th>$L_{\text{actual}}$</th>
<th>$L_{\text{random}}$</th>
<th>$C_{\text{actual}}$</th>
<th>$C_{\text{random}}$</th>
</tr>
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<tbody>
<tr>
<td>Film actors</td>
<td>3.65</td>
<td>2.99</td>
<td>0.79</td>
<td>0.00027</td>
</tr>
<tr>
<td>Power grid</td>
<td>18.7</td>
<td>12.4</td>
<td>0.080</td>
<td>0.005</td>
</tr>
<tr>
<td>C. elegans</td>
<td>2.65</td>
<td>2.25</td>
<td>0.28</td>
<td>0.05</td>
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Characteristic path length $L$ and clustering coefficient $C$ for three real networks, compared to random graphs with the same number of vertices ($n$) and average number of edges per vertex ($k$). (Actors: $n = 225,226, k = 61$. Power grid: $n = 4,941, k = 2.67$. C. elegans: $n = 282, k = 14$.) The graphs are defined as follows. Two actors are joined by an edge if they have acted in a film together. We restrict attention to the giant connected component of this graph, which includes $\sim 90\%$ of all actors listed in the Internet Movie Database (available at http://us.imdb.com), as of April 1997. For the power grid, vertices represent generators, transformers and substations, and edges represent high-voltage transmission lines between them. For C. elegans, an edge joins two neurons if they are connected by either a synapse or a gap junction. We treat all edges as undirected and unweighted, and all vertices as identical, recognizing that these are crude approximations. All three networks show the small-world phenomenon: $L \gg L_{\text{random}}$ but $C \gg C_{\text{random}}$. 
So far, we have just characterised **small-world networks** as those which have a high clustering coefficient, but a low characteristic path length.

**Scale-free networks** are a refinement of this, where additionally the degree of the nodes must follow a power-law (as in social networks).

But, the *rewiring the lattice* method does not result in a power-law distribution of connectivity.
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But, the *rewiring the lattice* method does not result in a power-law distribution of connectivity.

Alternative approach: Preferential Attachment

Preferential attachment means that the more connected a node is, the more likely it is to receive new links.

Intuitively:

- When a new person moves to the area, there’s a higher chance that they will meet people who are already well known.
- When a new website is set up, it’s more likely to link to a well known site (Google, Wikipedia etc.) than a page not many people know.

We thus have a positive feedback cycle, where initial random variations (e.g. one node beginning with slightly more links) are automatically reinforced.

This *magnifies the differences*. 
Preferential Attachment

1. Begin with a small network of $m_0$ vertices, such that $m_0 \geq 2$ and the degree of each edge is at least 1.

2. Add a new vertex with $m$ edges, where $m \leq m_0$ each that links to a different vertex already in the system, selected according to the following scheme:
   
   a. Choose vertex $i$ to connect to with probability $\Pi(k_i)$, where
   
      $\Pi(k_i) = \frac{k_i}{\sum_j k_j}$
   
   3. Repeat from step 2 until finished.
Building Small World Networks: Method 2

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   2. where $\Pi(k_i) = \frac{k_i}{\sum_j k_j}$

3. Repeat from step 2 until finished.

This leads to a **scale free network**, where the probability of a node having $k$ connections, $P(k) \sim k^{-3}$. 
Further Reading
