Introduction to Natural Computation

Lecture 17

Propagation Time in Networks

Alberto Moraglio
This Lecture

- Network problems
- Processes from nature
- Fun with probabilities
- Understanding randomness
- Beautiful theory
Use propagation to identify the youngest person in class.
Assume a graph with $n$ nodes.

Initially one node is informed.

In each round each informed node informs a neighbor with probability $p$.

The decisions for all neighbors are independent.
A Model of Propagation

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Question

How long does it take in expectation until all nodes are informed?
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**Question**

How long does it take in expectation until all nodes are informed?

**Applications**

- very simple model of disease spreading
- communication in wireless ad-hoc networks
- analysis of natural computation algorithms
Let us look at a chain of $n + 1$ nodes.
Expected Times

How long does it take on average until a neighbor on the chain gets informed?

A random variable is a variable whose value results from the measurement of some type of random process.

**Expectation**

Let $X$ be a random variable. Then

$$\sum_{x} x \cdot \text{Prob}(X = x)$$

is called expectation of $X$, written $E(X)$. 
Expected Times

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**Expectation**

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$$

is called expectation of $X$, written $E(X)$.

Example: what is the expectation of a fair 6-sided die?
What is the expected time until you roll a 6?
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Assume I have rolled a fair die four times without getting a 6. How many more times do I need to roll in expectation until I get a 6?
Example: Dice

What is the expected time until you roll a 6?

Assume I have rolled a fair die four times without getting a 6. How many more times do I need to roll in expectation until I get a 6?

What is the expected number of sixes if I roll 6 times?
Example: Dice

What is the expected time until you roll a 6?

Assume I have rolled a fair die four times without getting a 6. How many more times do I need to roll in expectation until I get a 6?

What is the expected number of sixes if I roll 6 times?

What is the probability that I will get at least one 6 if I roll 6 times?
Probability $1/6$, expected waiting time $6$. Coincidence?
More Generally . . .

Probability $1/6$, expected waiting time 6. Coincidence?

**Lemma**

Assume we repeat a random experiment that is successful in each time step with probability $p$, independent from other time steps. Then the expected time until the first success happens is $1/p$. 
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**Lemma**

Assume we repeat a random experiment that is successful in each time step with probability $p$, independent from other time steps. Then the expected time until the first success happens is $1/p$.

Math-speak: *geometric distribution* with parameter $p$. 
Chain
Let $T_i$ be the random time until $v_i$ gets informed, assuming $v_{i-1}$ is. We already know $E(T_i) = 1/p$. 
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Can we simply add all $E(T_i)$’s to compute $E(T)$?
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Yes, we can.

For any two (independent or not independent) random variables $X, Y$

$$E(X + Y) = E(X) + E(Y).$$
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**Theorem**

*The expected propagation time on a chain with $n + 1$ nodes is $n/p$.***
Consider situation after an \textit{epoch} of \( t := \ln(2n)/p \) time steps.
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What's the probability that we have not informed all nodes after one epoch?
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Probability that one specific outer node not informed:

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(1 - p)^t \leq \exp(-pt) = \exp(-\ln(2n)) = \frac{1}{2n}
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Probability that at least one of $n$ outer nodes not informed:

$$\leq \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = n \cdot \frac{1}{2n} = \frac{1}{2}$$
Analysing the Hub

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So, probability that epoch ends with all nodes informed \( \geq 1 - \frac{1}{2} = \frac{1}{2} \).
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New random experiment: wait for successful epoch.
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Expected number of epochs is at most \( 1/\frac{1}{2} = 2 \).
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So, probability that epoch ends with all nodes informed $\geq 1 - \frac{1}{2} = \frac{1}{2}$.

New random experiment: wait for successful epoch. Expected number of epochs is at most $= 1/\frac{1}{2} = 2$.

Theorem

*The expected propagation time on a hub with $n$ nodes is at most $2 \cdot \ln(2n)/p$.**
Theorem

For a star graph with $b$ branches of length $d$ each, the expected propagation time is at most

$$\frac{13(d + \log b)}{p}.$$
What About Other Graphs?
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The diameter is the maximum length of a shortest path between two nodes $u$ and $v$, where the maximum is taken over all pairs $u, v$. The length of a path is given by the number of edges.

![Graph Diagram]

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**Theorem**

For every graph with $n$ nodes and diameter $D$, the expected propagation time is at most

$$\frac{13(D + \log n)}{p}.$$
Further Reading
