

## WORK IN PROGRESS -- LIABLE TO CHANGE

### Non-monotonic angle change as a vertex moves on a line:

Discovery of a necessary connection between two different continua: variation in location of a point and variation in size of an angle, easily discovered by non-mathematicians.

(And some surprisingly complex variants of a simple problem.)

What brain mechanisms are required for thinking about this?

Can current computers model/replicate them accurately?

If not, what sort of information processing mechanism can?

Does any theory of consciousness explain mathematical consciousness?

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**N.B. Aaron Sloman is responsible for any errors in this document.**

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[Jump to Contents](#)

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#### History of this paper (Installed 3 Oct 2017)

This is a "side-shoot" of an online video and web page originally used for an invited workshop talk at IJCAI August 2017 in Melbourne, presented remotely:

Why can't (current) machines reason like Euclid or even human toddlers?

(And many other intelligent animals)

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/ijcai-2017-cog.html>

I'll call that "**The workshop web page**" below.

After I mentioned the triangle-stretching example in a CS theory group seminar in Birmingham on 29th September 2017, Auke Booij pointed out the answer to a question I had left open. That prompted me to move the discussion of the example into this new document. This paper is

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/deform-triangle.html>

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/deform-triangle.pdf>

The PDF version may be slightly out of date, from time to time.

On 3rd Nov 2017 Diana Sofronieva heard me talk about this in seminar in Leeds University. The next day, at a conference in Leeds, she gave me a surprisingly complex answer to one of the questions I raised below, using the Apollonius construction in Euclidean geometry. Her answer is presented and discussed in a separate document created on on 17 Nov 2017. <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/apollonius.html>

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An incomplete draft paper on requirements for understanding cardinal and ordinal numbers that generally go unnoticed by researchers on mathematical cognition and its neural underpinnings can be found here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/cardinal-ordinal-numbers.html>

A closely related side-shoot of all these discussions is a speculation about missing mechanisms in AI, Neuroscience, and psychology:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html>

A Super-Turing Membrane Machine for Geometers  
(Also for toddlers, and other intelligent animals)

These are all part of the Turing-inspired [Meta-Morphogenesis project](#)

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### Updated:

22 Nov 2017 Changed heading and related details. 25 Nov 2017 clarified 'history'. Added contents.  
21 Nov 2017 Part moved out to new paper on [Apollonius construction](#).  
8 Nov 2017 Updated ([below](#)); 9-17 Nov 2017;

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## CONTENTS

- [CONTENTS \(This list\)](#)
- [History \(above\)](#)
- [Background](#)
- [What needs to be explained?](#)
- [Figure 1: Stretch-internal](#)
- [What if the vertex moves on a line outside the triangle?](#)
- [The deforming triangle example](#)
- [Figure 1: Stretch-internal \(repeated\)](#)
- [What if the triangle is tilted away from you? \[Added 15 Jun 2018\]](#)
- [NOTE on Pylyshyn's FINST Visual Indexing Theory](#)
- [A modified version of the triangle stretch theorem](#)
- [Figure 2: Stretch-external](#)
- [Auke Booij's Counter Example](#)
- [Figure 3: Stretch-obtuse](#)
- [Figure 4: Perpendiculars](#)
- [Introducing circles](#)

- [Figure 5: Circles](#)
  - [The required Super-Turing mechanism](#)
  - [Example: Circle and triangle problem](#)
  - [Note: Answer to question about triangle and circle](#)
  - [Note: Kenneth Craik](#)
  - [Note: Answer to question about triangle and circle](#)
  - [Relevance to theories of consciousness](#)
  - [Note on background to this paper](#)
  - [REFERENCES](#)
- 

## Background

(Partly repeating the IJCAI workshop web page,  
<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/ijcai-2017-cog.html>.)

### Updated 27 Nov 2017

Why talk about deforming triangles? Because I think there are deep, largely unnoticed, aspects of the ways human and non-human animal minds work that are closely connected with the mechanisms underlying important non-numerical mathematical discoveries by ancient mathematicians, i.e. topological and geometrical discoveries (which I've argued elsewhere are at the root of competences relating to cardinal and ordinal numbers, as opposed to spuriously similar pattern recognition capabilities).

It is not always remembered that for ancient mathematicians the axioms and postulates in Euclidean geometry were not arbitrarily chosen starting formulae from which conclusions could be derived using pure logic, in modern axiom-based mathematics: the ancient axioms were all [discoveries](#).

There were also important discoveries that were not included: for example the possibility of extending Euclidean geometry with the "neusis" construction, which was known to Archimedes and which made it easy to trisect an arbitrary triangle -- impossible in Euclidean geometry. For more information see: <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html>

I suggest that the mechanisms involved in thinking about what happens to angles of a triangle as it gets stretched by motion of one vertex relative to the other two were also available to ancient mathematicians, whether they thought of this example (explained below) or not.

Other mechanisms ancient mechanisms are good at topological categorisations, e.g. X and Y are connected/disconnected, X is contained in/overlaps with/outside Y, and more complex cases.

Those mechanisms, or closely related mechanisms, are also needed for intelligent perception and action in richly structured, extremely varied spatial environments, especially

I suspect that all the core mechanisms for making and using such discoveries were available in brains that had evolved in environments in which how things look, and what you can see, change as you move, or as objects move. In some animals those mechanisms were extended to enable consideration of structures and processes that are not actually occurring, but which might occur, whether caused by the perceiver or not. (I have previously labelled these, and other examples of

perceived possibilities for change in the environment, "proto-affordances", extending Gibson's theory of affordances [Gibson\(1979\).](#))

The reflective ability to notice uses of such abilities, think about them, discuss them, and understand why they work, probably evolved only in humans, though the perception and use of proto-affordances and other sorts of affordances seems to occur in many intelligent species.

Many researchers, especially AI vision and robotics researchers, and some psychologists and neuroscientists assume that spatial percepts necessarily require use of numerical measures, e.g. of distance, width, height, speed of motion, angles, areas, etc., or at least probability distributions over numerical measures.

However, biological sensors are mostly not good at precise numerical measurement, though some of them are good at detecting changes, or direction of change (increasing/decreasing) in physical relationships across space or time -- often using *partial orderings*, e.g. A is further than B, A is further than B by more than C is further than D, and many more, without being able to tell whether A is further from C than from D.

Such competences with partial orderings allow use of descriptions of structures and processes that are [accurate](#) but not [precise](#), and which therefore, when they are available, suffice for control of actions without complex numerical computations of probabilities, now widely used in AI, including robotics. (For some simple examples explaining the power of partial orderings and topological relationships, see [Sloman\(2007-14\).](#))

I suspect that at some (remote?) future date we'll understand how those non-metrical spatial detection mechanisms can contribute to rich and extremely useful structural and ordering information used by our remote ancestors and some other intelligent species interacting with complex structures and processes, including other agents -- offspring, mates, friends, live food, and foes, and perceived structures and materials used for many purposes, including feeding, building nests, and in some cases making tools.

I suggest that the current concern of scientists and engineers with [numerical](#) measures and [numerical](#) control mechanisms, combined with use of [probabilities](#) to compensate for limitations of accuracy and completeness in available measures in studying and modelling various aspects of natural intelligence, is more a product of the prejudices currently built into our scientific and engineering educational practices than requirements for working mechanisms -- natural or artificial.

Biological optical sensors in birds and mammals (e.g. as described in [https://en.wikipedia.org/wiki/Visual\\_system#Retina](https://en.wikipedia.org/wiki/Visual_system#Retina)) seem to have far more complex functionality built in to their low level design than any human-engineered video camera, even those with the complexity of "Three-CCD" cameras, summarised in [https://en.wikipedia.org/wiki/Three-CCD\\_camera](https://en.wikipedia.org/wiki/Three-CCD_camera).

Perhaps in future we'll learn more by finding out what all that complexity is actually used for than by building models based on assumptions about what it must be used for!

Some of the hypothesised topological and geometrical reasoning abilities are analysed below, and will be discussed further in a separate document exploring the idea of a Super Turing machine, mentioned briefly below. Many more examples of mathematical impossibilities and necessities are presented in

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/impossible.html>

Later I'll discuss the problem of specifying mechanisms that can perform these tasks. Since the relationships found involve extreme modalities **necessity** and **impossibility**, they cannot be inferred from empirical evidence: they do not involve probabilities, for example, as explained in my DPhil Thesis [Sloman 1962](#), defending claims in [Kant\(1781\)](#).

**Note:**

I think related topological mechanisms were originally required for (gradual) development and use of concepts of cardinal and ordinal number, both based on implicit understanding of properties of 1-1 correlations, long before anyone thought of modern axioms for natural numbers. Some relevant ideas are in [chapter 8](#) of [Sloman 1978](#), and [Sloman\(2016\)](#) (in the Section: "What about arithmetic?" Pages 10-13), expanded in this draft document (Nov 2017): <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/cardinal-ordinal-numbers.html>

If those conjectures are correct then all claims that cardinal or ordinal number concepts are innate are false. What seems to be innate (and shared across several species) is merely a superficially related template-based pattern recognition ability that for small numbers produces what look like answers to questions about cardinality. [Links/references to be added.]

This is one of several online discussions of human (and in some cases non-human) abilities to perceive impossibilities and necessary connections that seem to have evolved in the mechanisms required for dealing with spatial affordances, and later formed the basis of number competences. Those abilities were used in ancient mathematical discoveries long before the development of modern logic and formal systems. Some aspects of those ancient mathematical competences also seem to characterise abilities of pre-verbal toddlers and other animals, illustrated in videos linked from my IJCAI 2017 AGA Workshop web page: <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/ijcai-2017-cog.html>.

The currently dominant strategy in AI, psychology, and neuroscience for explaining natural intelligence postulates use of statistical evidence to build up re-usable information about probabilities. In contrast, for many years I have been collecting examples suggesting that that is a completely misguided theory of natural intelligence, and a poor basis for robot intelligence. (My work is partly inspired by ideas in [Kant\(1781\)](#).)

Some of the evidence comes from close observations (preferably video recordings) of pre-verbal humans and other animals and from analysis of requirements for intelligent decision making and the kinds of information (e.g. about topological structures and partial orderings rather than numerical measurements) readily available in the environment, discussed in [Sloman\(2007-14\)](#) and related online documents.

For example, you can safely walk through a doorway by aiming for a location between the left and right doorposts and well clear of both, without having either precise measurements of the locations and distances of the doorposts or probability distributions over numerical values.

Many examples can be found in spontaneous, often surprising, reactions to objects or opportunities in the environment, in young children and other animals. Many of these are not repeatable, but that does nothing to diminish their relevance to deep science in which the primary advances come from discovery and explanations of what is *possible*, rather than discovery of *laws* or *regularities*, as

explained in [Chapter 2](#) of [Sloman 1978](#). (This contradicts Popper's widely, but mistakenly, accepted claim that all scientific statements should be empirically falsifiable.)

For several decades I have been collecting examples of ancient mathematical discoveries in geometry and topology, and some previously apparently unnoticed discoveries (like the triangle stretch discovery below) that seem to be linked to achievements of pre-verbal humans and other intelligent animals, e.g. when using perceived spatial structures and relationships to select and control actions. How such abilities, not specified in the genome, can depend on and be related to more general and abstract features of the genome is the topic of another document: The meta-configured genome (based on work with Jackie Chappell):  
<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-configured-genome.html>

In contrast, most published research on human and non-human mathematical intelligence focuses on [numerical](#) competences, ignoring the possibility that those competences were built on older, more fundamental abilities to reason about spatial or spatio-temporal structures and relationships, including features of structural changes from which ideas about one-to-one correspondences (bijections) emerged, later followed by spatial reasoning abilities used in discoveries and theories concerned with numbers of various types. Centuries (or millennia) later, those were supplemented by use of much more formal mechanisms known to and understood by professional mathematicians, but irrelevant to most of those who regularly use number concepts.

The developments I am trying to understand must have occurred long before the advent of modern formal methods in mathematics. Numerical competences are not the main topic here, however. This paper is about older, deeper competences.

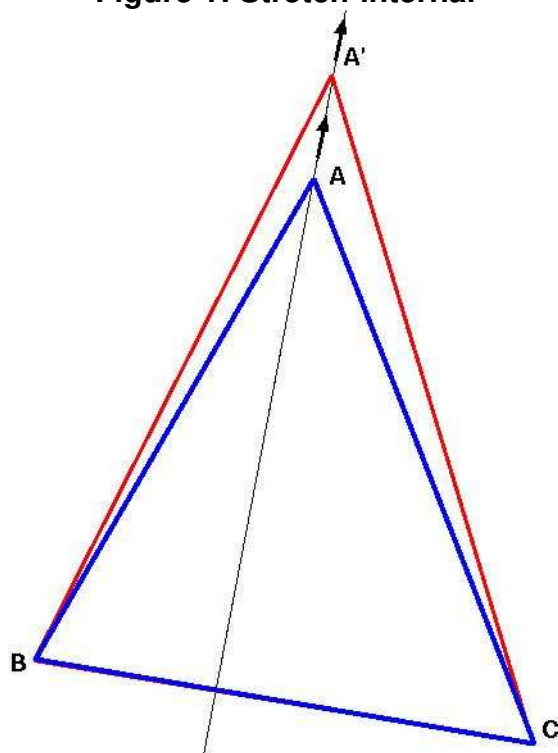
Any theory of intelligence or design for intelligent machines that does not take account of these ancient spatial reasoning components of natural intelligence will fail to explain commonplace abilities of humans and other intelligent animals and their computer models will fail to replicate important aspects of natural intelligence, including failing to model processes of mathematical discovery, even if they produce results that look superficially similar, achieved by very different mechanisms.

## **What needs to be explained?**

There have been many attempts to understand brain mechanisms supporting mathematical abilities. Unfortunately, most start from an inadequate characterisation of what needs to be explained (a topic for another time). In a talk at the [International Conference on Cognitive Modelling \(ICCM\)](#) in July 2017, I decided to use a new problem that, as far as I know, does not figure in mathematical text books, namely

"If ABC is a triangle of any shape, and the vertex A is moved in a straight line away from the opposite side BC, what happens to the size of the angle at A?"

**Figure 1: Stretch-internal**



Possible forms of reasoning are discussed in [the main presentation](#).  
The example is also discussed in the video for the IJCAI talk: [\[VIDEO LINK\]](#)

Everyone I talked to (adult academics of various sorts) seemed to find it obvious that the angle at A must steadily decrease in size as A moves further from the opposite side -- no matter how far it has moved and how small the angle already is.

I use this question and the general form of the responses obtained, to motivate questions about the nature of the perception and reasoning mechanisms that allow such a discovery to be made and to be seen to express a necessary truth (provided that the line on which A moves passes between B and C, i.e. crosses the opposite side of the triangle). That raised interesting questions about the cognitive machinery required to grasp such a necessary truth, including questions discussed by Immanuel Kant in [Kant\(1781\)](#).

Insofar as people notice that the angle **necessarily** grows smaller as the distance from the opposite side increases, and do so without having done any measurements or physical experiments or collected statistics (except perhaps implicitly and unwittingly over sub-ranges of shape change), they cannot be reporting an empirical generalisation.

There's an interesting discussion by Francesco Beccuti of what happens in the limit, in [Beccuti\(2018\)](#).

Instead they are somehow conscious that there is a **necessary** connection between the change of length and the change of angle. Consciousness of such necessities is a characteristic feature of mathematical discoveries (as Kant pointed out). So any theory of consciousness that does not account for mathematical consciousness (including consciousness of mathematical necessities and impossibilities) must be inadequate as a general theory of consciousness: which applies to almost everything I have read about consciousness, except in Kant's work. (Mathematical theories of

consciousness are easily found online, but the examples I have encountered do not explain, or even describe, mathematical consciousness!)

## **What if the vertex moves on a line outside the triangle?**

Returning to the observation that size of an angle at a vertex shrinks as the vertex moves further from the opposite side, things are not so clear if A moves further away from the opposite side along a straight line that does **not** pass between B and C, instead intersecting the base BC outside the triangle, as in [Figure 2](#) below.

During a presentation to the Theoretical Computer Science group in my department on 29th Sep 2017 based on the material in [the workshop web page](#), I mentioned these examples of reasoning about consequences of triangle deformations, including the conjecture that motion of a vertex away from the opposite side always caused the angle at the vertex to shrink in size. This was challenged by Auke Booij.

During a subsequent email discussion he pointed out my omission: I had not thought about some of the details, presented below, using diagrams depicting shape-changing triangles. Later it turned out that this problem had extraordinary mathematical complexity linked to a problem and a solution discovered by ancient mathematicians -- about which I first learnt from Diana Sofronieva after she heard me talk about this topic in Leeds University in November 2017.

This suggests to me that requirements for perception and reasoning about spatial structures in the physical environment led to evolution of increasingly sophisticated topological and geometrical (mostly non-numerical) forms of information processing in biological organisms eventually producing forms of perception and reasoning that used abilities to consider unperceived, but possible, spatial changes with structural features that increased in sophistication over many generations long before the knowledge acquired was organised by ancient mathematicians.

By the time the first humans existed the mechanisms that had evolved to meet a wide collection of practical perception and reasoning problems, including those presented in [Sloman\(2007-14\)](#), already had the power also to support hypothetical, theoretical reasoning about surprisingly complex changing structures and processes. I suggest that these laid the foundations for later discoveries presented in Euclid's *Elements*.

## **Why the stretched triangle example?**

The example of reasoning about a stretched triangle is worth considering as one among many windows into unobvious sophistication in those animal competences. The initial observation about the effect of stretching a triangle seems so obvious and simple that it is (at least to me) surprising how quickly it leads into a sophisticated problem in Geometry, pointed out to me by Diana Sofronieva (henceforth Diana).

I'll start with the observations of Auke Booij who had previously helped me notice some unobvious complexities in other geometric problems). A [separate document](#) presents the analysis provided by Diana, who identified a connection with the problem of Apollonius, which I had not previously encountered.



## The deforming triangle example

To illustrate the sort of discovery process enabled by the postulated (but still under-specified) "membrane" system, I gave examples related to [Figure 1](#) above (repeated below for convenience).

What will happen if you start with a triangle, like the blue triangle ABC in the figure, and move the top vertex (corner), A, away from the opposite side (the base of the triangle) along a line going through the base. (Other traversal lines are considered below.) The red triangle in the figure, with a new vertex A', illustrates one of the possible new locations to which the triangle could be moved. Try to formulate an answer that is independent of the size, shape, and orientation of the triangle.

You probably find it obvious that the angle at the moving vertex will continually decrease in size as the vertex moves further from the base, the side opposite it in the triangle. How can one know that?

Most people asked the question about the figure on the left seem to find the answer obvious, saying that as the vertex A moves further from BC the angle at A must get smaller. It is not at all clear what form of reasoning they are using -- nor what their brains are doing.

### Two mathematical continua

The people I have asked don't seem to be aware that in answering the question they have identified [two mathematical continua](#) (the continuum of locations of the moving vertex on the line through the triangle, and the continuum of angle sizes for the vertex as it moves), and a systematic (monotonic) relationship between them.

Each continuum is involved in the motion of the vertex, and each can be attended to separately. Everyone I have asked also finds it obvious that the two are rigidly connected: changes in the location cannot occur without changes in the angle size. This is not an explicit mapping between individual vertex locations and individual angle sizes, rather it is a rigid relationship between directions of change in the two: each direction of motion of the vertex is necessarily connected with one of the directions of change of size of the angle at the vertex. What sort of reasoning mechanism can make such a discovery so quickly will be discussed later in connection with a proposed new non-digital computing machine.

I have so far not asked anyone whether returning the vertex to a previous location will necessarily restore the previous angle size. Anyone with knowledge of Euclidean geometry will find the answer obvious because the initial and restored triangles must be congruent, insofar as restoring the (relative) location will produce sides of the same lengths as the initial triangle. But that background knowledge is not required for answering the question about direction of change.

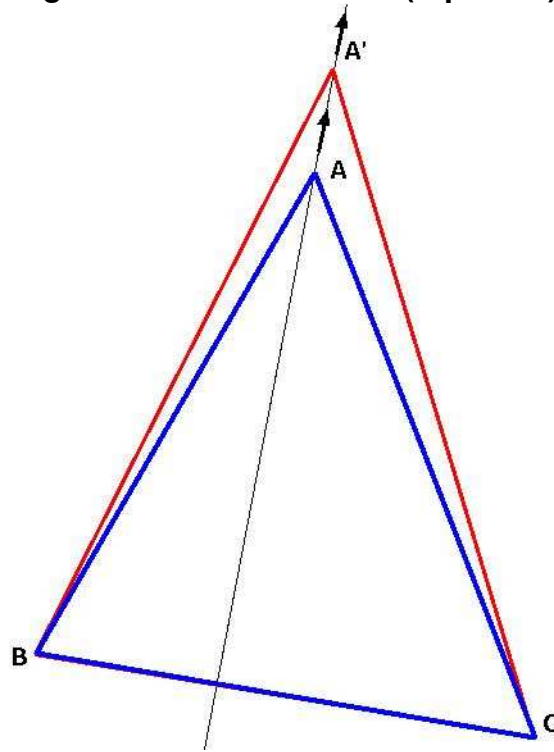
In general, however, two processes may have necessarily linked [directions](#) of change without having necessarily linked [locations](#) in the space of possibilities. (E.g. there could be "hysteresis" i.e. dependence of state on previous history. I'll leave that possibility for discussion another time.)

### Moving on a different line

The situation changes if the direction of motion of the vertex changes so that it moves on a line that does not intersect the base of the triangle (the side opposite the moving vertex)? This case was discussed briefly in the workshop web page, leading up to the idea of the membrane mechanism. Below I'll discuss alternative lines through A along which A can move, and ask what difference the line makes to how the size of the angle at A varies. This apparently simple question uncovered (with help from Auke, then Diana) a surprising "bag of worms", extending requirements for the

Super-Turing machine, for which ideas are under development here:  
<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html>

**Figure 1: Stretch-internal (repeated)**



I claim, but will not argue here, that all of these examples are related to mechanisms that form part of everyday perception of spatial structures and affordances in humans (including infants) and other intelligent animals [Sloman\(2007-14\)](#). In particular, changes of relative size or angle as you move towards or away from something, or move one thing relative to another, can produce perceived changes in visible relationships between edges, corners and other features of objects. I argue in [Sloman\(2007-14\)](#) that this plays a far more important role in spatial intelligence than has been generally noticed, and if implemented in robots could simplify some of their control problems.

Evolution also produced mechanisms for [envisaging](#) or [imagining](#) such changes when they are not actually occurring, without doing any numerical calculations of values of point coordinates or angles or areas, as a typical current robot or other AI system would have to do. (Future AI systems may be more like humans as a result of this work.)

An example is looking at two blobs of colour and deciding whether one is larger than another by [visualising](#) one sliding over the other rather than doing measurements and calculations.

There have been attempts to give machines related competences, e.g. machines that have mechanisms for painting an image into a spatial data-structure. Those may be fairly close to some of the cases considered below, but without the meta-cognitive abilities to notice and reason about consequences of such spatial operations.

Traversal in a visual field can arise in various contexts, including monitoring of an object moved by the perceiver, perception of an object that is moving independently of the perceiver, monitoring of effects of changes of view direction (e.g. saccades), consideration of a possible, visually imagined motion, or motion of the perceiver.

In some cases the object seen to be moving is fixated, so that everything else moves in the visual field. In other cases an object moves without being fixated, so that it moves across the visual field. We'll ignore those differences here, though they are important for a complete account of visual cognition.

These features of spatial perception are important in previously unnoticed ways, for a complete account of mathematical cognition. Consideration of unnoticed details in many different geometrical examples can help to draw attention to unnoticed features of much broader classes of perception, reasoning, planning, and control of actions.

## **What if the triangle is tilted away from you?**

A quite different question arises if the triangle is drawn on a vertical plane surface, facing you, with the base of the triangle, BC, horizontal, e.g. close to the bottom edge of the surface, perhaps resting on a table or on the floor in front of you, and is tilted backwards. Assume that the base of the triangle remains close to the fixed horizontal edge of the plane, while the rest of the plane moves away from you, while remaining in the plane as it tilts. (Easier to demonstrate than to describe.)

### **Questions:**

What will happen to the appearance of the triangle in that case? In particular, what will happen to the perceived angle at vertex A as the tilting causes A to move away from you? Will the angle in your visual field increase in size, remain the same, or decrease? How do you know?

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## **NOTE on Pylyshyn's FINST Visual Indexing Theory**

The mechanism required here may be related to "The FINST Visual Indexing Theory" of Zenon Pylyshyn and colleagues around 1990, summarised in <http://rucss.rutgers.edu/val-current-research/23-labs/val/231-the-finst-visual-indexing-theory> I don't know whether Pylyshyn attempted to use his mechanism to explain geometrical discoveries made by ancient mathematicians (and young children, intelligent non-human animals, etc.). I think the mechanism was mainly intended to allow internal processes to refer to objects of attention, including moving objects. The ideas are relevant to my topic, though at present I only dimly recall the details. (I think Pylyshyn and I have discussed some of these issues in the very distant past.)

The variety and prevalence of perceived changes involving two or more related changes of structures or relationships when intelligent organisms perceive or produce changes in the environment, some of them produced by other intelligent organisms, does not explain **how** evolution produced these detection mechanisms, but it does explain why they should be preserved by reproductive and other processes after they first become available, and why their absence in current AI spatial perception mechanisms is a very serious deficiency.

### **NOTE:**

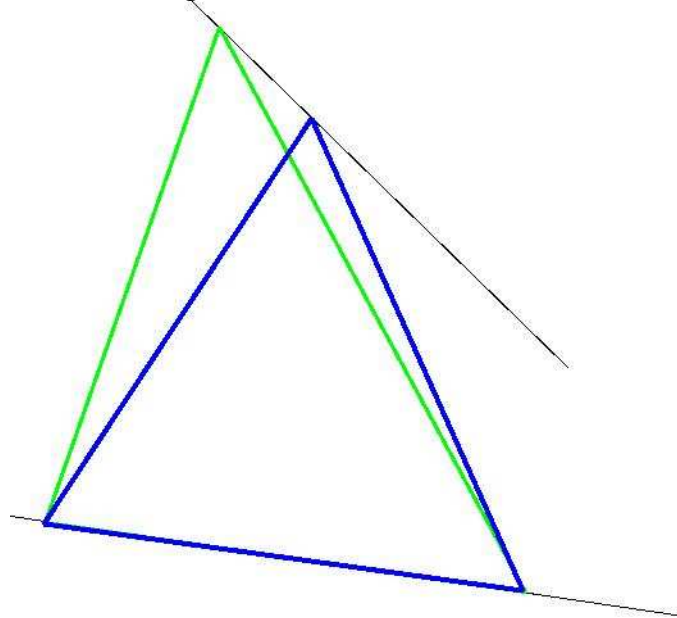
This problem about how angles change as one corner of a triangle moves is loosely related to ways of reasoning about how the area of a triangle changes as the triangle is deformed (the "area stretch" theorems) explored in [another document](#).

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## A modified version of the triangle stretch theorem

Now consider what difference it makes if we modify [Figure 1](#), above, so that the line along which the top vertex moves intersects the opposite side (the base) of the triangle **outside** the triangle as in [Figure 2](#), below. What will happen to the top angle if the vertex moves upwards (and to the left) along the new line? Does the size of the moving vertex necessarily get smaller, as in Figure 1?

**Figure 2: Stretch-external**



Vertex moves upward along a line through base extended.

The new type of case is harder to reason about. In [Figure 2](#) neither of the top angles is contained in the other. The new example requires a notion of [comparison of size of two angles](#) that is independent of orientation of the angles, and whether one angle can obviously be moved so as to be contained in another. In the workshop web page, finding a proof for the new configuration was left as an exercise for the reader, and as stimulation to investigate the differences between the two cases.

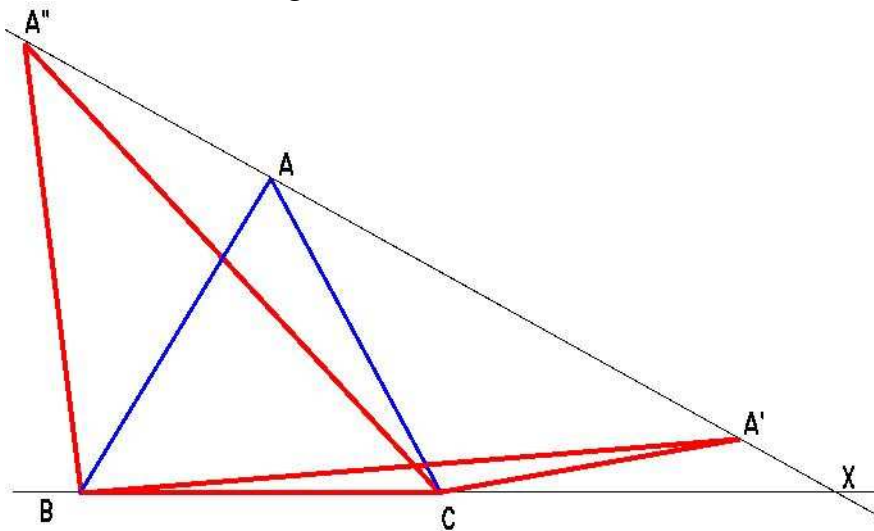
I turn now to an objection raised when I conjectured that in [Figure 2](#) it is also the case that as the vertex moves up along the line intersecting the base of the triangle (extended here) the angle at the vertex constantly gets smaller.

### **Auke Booijs Counter Example**

During the presentation on 29th Sept, I suggested that the size of the top angle in [Figure 2](#) would always decrease while the vertex moves "upwards" along a line through the base extended. Auke pointed out that it depends where the vertex starts on the line, and clarified this by email later. [Figure 3](#), below, illustrates his objection.

If the vertex starts far enough to the right on the line of motion, e.g. at  $A'$ , close to where the line of motion meets the extension of the base of the triangle, at location  $X$  in [Figure 3](#), below, the angle at  $A'$  is smaller than the angle at  $A$ , and as  $A'$  moves closer to the intersection at  $X$ , the angle at  $A'$  must get smaller. (Why?)

**Figure 3: Stretch-obtuse**



Consider the red triangle whose shape is represented by  $A'BC$ , in Figure 3 above, and how it might be deformed into red triangle  $A''BC$  by moving the vertex at  $A'$  in a straight line towards  $A''$ . At an intermediate stage the triangle's location is shown in blue as triangle  $ABC$ . (Compare the blue triangles in [Figure 1](#) and [Figure 2](#) above.) There are some "implicit" discontinuities in the trajectory depending on

- whether  $A'$  is on the extended line  $BC$ , at location  $X$ . The angle there must be 0 degrees.
- whether  $A'$  is to the right of the line through  $C$  perpendicular to  $BC$ , for example the location  $A'$  shown in Figure 3.
- whether  $A'$  is on the line through  $C$  perpendicular to  $BC$ , not drawn on the diagram,
- whether  $A'$  is between the lines perpendicular to  $BC$  at  $B$  and at  $C$ , an example of which is shown at location  $A$  in the diagram,
- whether  $A'$  is on the line through  $B$  perpendicular to  $BC$ , not shown,
- whether  $A'$  is to the left of the line through  $B$  perpendicular to  $BC$ , e.g. the location shown as  $A''$ .
- Some readers may also by now have considered the perpendicular bisector of line  $BC$ , introducing three new sub-cases, namely, whether the location of  $A$  where the angle is largest is on either side of the bisector or on the bisector.

**Figure 4: Perpendiculars**

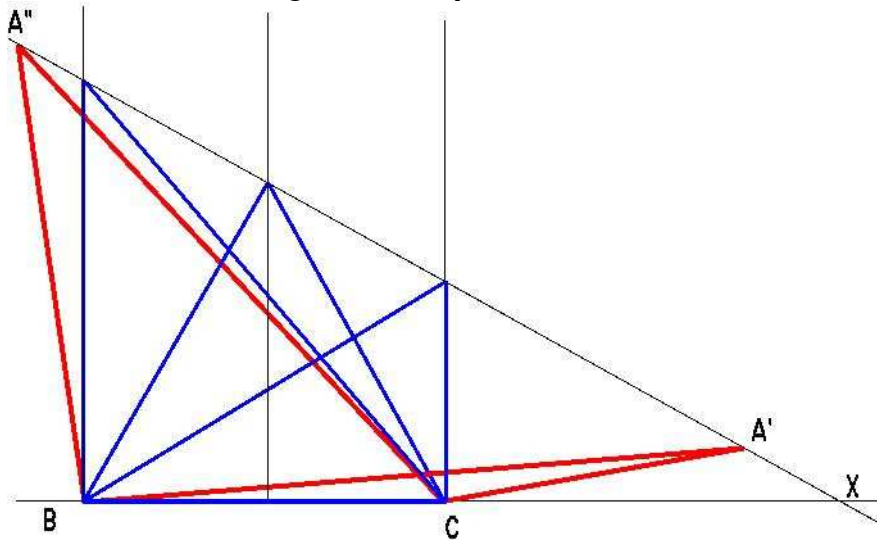


Figure 4 shows how lines perpendicular to BC can divide up the sloping line. A mathematician may investigate whether one of those perpendiculars defines a location of maximum angle for the moving vertex, and then conclude that perpendiculars to BC do not help to identify a point at which the vertex A is maximal. However this is not at all obvious. Is there any other way to identify a point at which the angle A is largest?

Ingenious readers may be able to think of more subdivisions between possible types of location of the moving vertex, arising out of the relationships between items involved in the initial problem formulation. For example, another subdivision, involves circles.

## **Introducing circles**

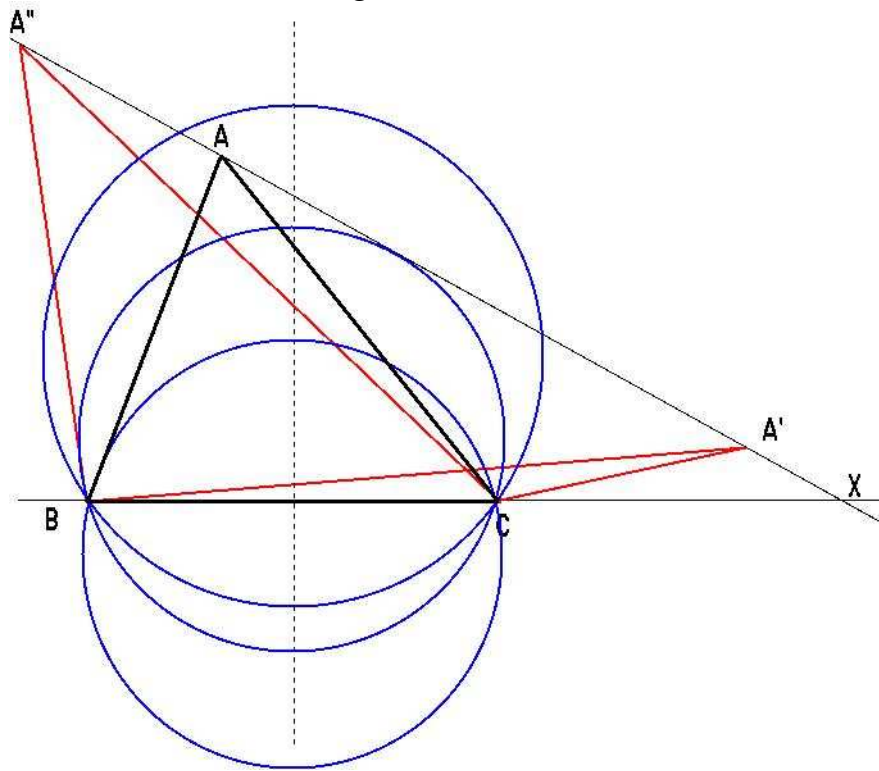
**Added/updated 8-9 Nov 2017**

At a seminar in Leeds on 3rd Nov 2017 I presented Figures 1 to 4 above as examples of geometric reasoning that are different from logical or arithmetical reasoning, and posed the problem of finding the location along the line  $A'A''$  at which angle BAC is largest. A philosophy student at the seminar, Diana Sofronieva thought about the problem and the next morning gave me a very helpful solution. My discussion of circles below is based on her solution. The solution depends on a problem in Euclidean geometry, Apollonius' problem (which has several different forms), which I had never previously encountered. Some of the key ideas are summarised below, but the detailed analysis is somewhat specialised, so it has been moved to a separate file

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/apollonius.html>



Figure 5: Circles



There are no circles visible in the earlier diagrams (before Figure 5), yet the Euclidean plane allows infinitely many circles to exist. In particular, there are infinitely many different circles passing through the two points B and C, the bottom vertices of triangle ABC in [Figure 5](#) above. It turns out, surprisingly, that the answer to the question where the angle size is maximal depends on circles!

Three examples are the blue circles shown. Each circle passes through points B and C, and because the centre of each circle must be the same distance from B and from C, the centres must all be on the perpendicular bisector of the line BC, shown in Figure 5 as a vertical black dashed line.

### Exercise for the reader

Added 10 Nov 2017

Think about circles with centres at various locations on that vertical line, with various diameters, and see if you can divide the circles into different categories in terms of how they relate to points B and C and the line  $XA''$ . For points consider whether they are outside the circle, or on the circle, or inside the circle. For an infinite line consider whether it goes through the interior of the circle, or merely touches the circle as a tangent, or has no point in common with the circle.

What brain mechanisms make it possible to engage in such thinking?

Could you design a robot that could have such thoughts -- visualising various configurations and noticing how certain changes inevitably produce other changes, and also discovering impossibilities, like the impossibility discussed as a ["Transitional case"](#) below?

Contemplating [Fig 5](#) should convince you that there are infinitely many circles that pass through the points B and C and are entirely [below](#) the line from  $A'$  to  $A''$ , and share no point with that line, as illustrated by the lowest blue circle in [Figure 5](#). Likewise there are infinitely many circles through B

and C that pass **above** the line A'A", cutting it in two different points, as illustrated by the highest blue circle.

However, as the diameter of a circle through B and C shrinks or expands, while the centre moves up or down the perpendicular bisector of B and C, there must be one, and only one, circle through points B and C that has line A'A" as a **tangent**, i.e. exactly one circle that touches the line at exactly one location above the line through the base of the triangle, indicated by the middle circle in [Fig 5](#).

There is another, much larger, circle through points B and C, that touches the line A"X extended from below, far to the right of the points in the diagram. For details see [the paper on the Apollonius construction](#).

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## The required Super-Turing mechanism

[The IJCAI workshop presentation](#) mentioned some features of logical reasoning that enabled such reasoning to be replicated on a turing machine or digital computer, namely the reasoning can be based on manipulation of discrete structures that do not vary continuously, and which can be combined to form larger structures, e.g. by insertion into adjacent cells on a Turing Machine tape, or into a logical or algebraic string of atomic symbols.

But symbols on a Turing machine cannot be superimposed in the way that generates new geometrical sub-structures two shapes are superimposed, e.g. a circle and a triangle, or a straight line and a triangle. A Turing machine does not allow two or more symbols to be superimposed: each has a part of the tape to itself.

### Example: Circle and triangle problem

For example you can combine a circle and a triangle on a planar surface, and by sliding them around you can produce different numbers of points common to the triangle and the circle, including no points if they do not overlap, a maximum of six points, and intermediate numbers. (Exercise for the reader: apart from 0 and 6 points common to the circle and triangle, what other numbers are possible, and in what configurations?)

Below, I'll try to show how exploration of the initial hypothesis that vertex size always decreases steadily with distance from the opposite side of a triangle, can lead to unobvious counter-examples as a result of unanticipated consequences of interactions between parts of the diagram: changing some geometric relationships can cause surprising new relationships, or constraints, to emerge.

These examples help to extend the requirements specification for a conjectured "Super-turing membrane mechanism" for spatial reasoning, explored in a parallel document:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html>

I conjecture that biological evolution originally produced such a mechanism to support processing of visual information of mobile animals, including perception of changing affordances, as discussed in [Sloman\(2007-14\)](#), and from time to time added increasing complexity to the mechanisms, as the organisms became more sophisticated in their needs, physical capabilities, and uses of spatial information.

The changes included forms of meta-cognition, and meta-meta-cognition, including what could be called "experimental meta-cognition" namely consideration of how contents of cognition (percepts) would change under various conditions.

It is worth noting that humans born blind may have a-modal spatial reasoning mechanisms that originally evolved as parts of visual perception then visual reasoning, but became amodally accessible so that they could also be related to tactile, haptic and auditory spatial perception. Perhaps also vestibular perception of self-motion?

I think such a mechanism was used by ancient mathematicians (unwittingly of course). Parts of the mechanism that evolved long before humans, are already present in human toddlers and other intelligent animals, but without the meta-cognitive additions available to adult humans, especially mature mathematicians, which allow the spatial reasoning processes to be attended to, recorded for later access, and eventually facilitated teaching and learning processes in which one individual thinks about, refers to and comments on the thinking processes of another.

Understanding the nature of those ancient (mostly unconscious) mathematical competences, and their evolutionary and developmental histories is an important part of understanding what animal minds are and how they work (as I think [Kant](#) understood, though I shall not argue for that here).

It is also an important part of the project to replicate intelligence of humans and other animals in machines. It is not yet clear to me whether virtual machines implemented on digital computers can support implementation of this competence, or whether another kind of machine, tentatively labelled a "Super Turing Membrane machine" can do the job.

#### **Note: Kenneth Craik**

There is an extraordinarily perceptive (partly prophetic), though incomplete, discussion of how brains might represent and reason about geometric figures and their relationships in [Craik \(1943\)](#) before digital computers were available, in Chapter 6, section headed "Abstraction and brain mechanisms".

Craik also anticipated some of the arguments in [Sloman\(2007-14\)](#) concerning the relative importance of partial orderings vs absolute measures for intelligent perception and action. (It's possible that I was influenced by having read Craik's book around 1967, before I encountered AI.)

#### **Note: Answer to question about triangle and circle**

The number of points common to a triangle and a circle can vary between 0 and 6. Some of them are 'touching' points, others overlap points. What sort of machine can discover (a) that there are exactly 7 possibilities and that all of them can occur, some of the numbers in more than one way.

Would inspecting lots of randomly generated pictures of a circle and a triangle be a good strategy? If not, why not?

#### **Relevance to theories of consciousness**

Any theory of intelligence, or theory of consciousness, that does not address this ancient kind of mathematical intelligence or associated types of consciousness of [impossibility](#) or [necessity](#) cannot be a serious candidate for a general theory of human or animal consciousness, including consciousness of pre-verbal toddlers, nest-building birds, hunting mammals, octopuses, cetaceans, orangutans, elephants, and ancient mathematicians!

Eventually I hope that the evidence collected in this and related documents will help us assemble new deep requirements for the information-processing mechanisms that explain those ancient discoveries and related aspects of natural intelligence, including perception of affordances, and reasoning about affordances.

Some of the material previously in the IJCAI-17 workshop web page(above) has been moved out into a first draft (incomplete) attempt to specify features of the information-processing machinery that evolution seems to have provided that so far has not been replicated in AI and has not been studied in psychology or neuroscience, which I have loosely labelled "The super-turing membrane machine", with requirements and speculative partial designs collected in this incomplete draft: <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html>

The remainder of this document focuses on a collection of increasingly complex spatial discoveries that can emerge from use of spatial imagination in idle reflection on possible distortions of a triangular shape. A slightly different problem, first drawn to my attention by Auke Booij, turns out to have a far more complex solution, found by Diana Sofronieva, and explained in a separate document. It raises issues about highly specialised cognitive abilities of ancient mathematicians whereas this document is about mathematical features of commonplace widely shared, but largely unnoticed, abilities.

## Note on background to this paper

The work reported here is part of a long term project to identify forms of virtual machinery with causal powers required to explain varieties of biological information processing. My earliest attempt was [Sloman\(1971\)](#), followed by [Sloman 1978](#), and various papers on vision, motivation, emotions, and other aspects of mind.

[Sloman and Chrisley\(2003\)](#) showed how certain kinds of virtual machinery could explain the ineffability and incomparability of forms of consciousness in different individuals. Further work on causal powers of virtual machine events and processes that in an important sense are not reducible to the physical mechanisms in which they are fully implemented are explained in [Sloman \(2013\)](#). A closely related viewpoint was presented in [Maley and Piccinini \(2013\)](#).

Gilbert Ryle's theory of consciousness as [polymorphous](#) is related to this, but I don't think he ever applied it to mathematical consciousness. However, the chapter on *imagination* in [Ryle \(1949\)](#) shows that he had come close to pre-inventing the idea of a virtual machine with causal powers.

A quick reading of <https://www.wired.com/story/new-math-untangles-the-mysterious-nature-of-causality-consciousness/> suggests that Erik Hoel has produced some related ideas, without noticing what has been learnt about phenomena in engineered virtual machinery that are not based on the mathematical mechanisms of noise reduction. Evolution is a deeply creative (*blindly* mathematically sophisticated) engineer, so far unmatched by human engineers in many respects, though overtaken in others, as explained in the theory of evolved construction kits [Sloman\(2017\)](#) (work in progress).

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### 21 Nov 2017

The section previously here concerned with properties of the Super Turing machine has now been moved into the paper discussing requirements for [the Super-Turing machine](#))

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<b>CONTENTS</b>	2
<b>Background</b>	
(Partly repeating the IJCAI workshop web page, <a href="http://www.cs.bham.ac.uk/research/projects/cogaff/misc/ijcai-2017-cog.html">http://www.cs.bham.ac.uk/research/projects/cogaff/misc/ijcai-2017-cog.html</a> .)	3
<b>What needs to be explained?</b>	6
<b>What if the vertex moves on a line outside the triangle?</b>	8
<b>The deforming triangle example</b>	9
<b>What if the triangle is tilted away from you?</b>	12
<b>NOTE on Pylyshyn's FINST Visual Indexing Theory</b>	12
<b>A modified version of the triangle stretch theorem</b>	13
<b>Auke Booij's Counter Example</b>	14
<b>Introducing circles Added/updated 8-9 Nov 2017</b>	16
<b>The required Super-Turing mechanism</b>	18
<b>Note on background to this paper</b>	20
<b>REFERENCES</b>	21