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Every
intelligent ghost
must contain
a machine

...
an information-processing machine

From Molecules to Mathematicians:

**How could evolution produce mathematicians
from a cloud of cosmic dust?**

What might Turing have done if he had lived longer?

**How might it have helped us understand biological evolution,
and how evolution produced mathematicians?**

Notes for a presentation to the Midlands Logic Seminar
http://midlandslogic.org.uk/2013_2014/Term1/Week5.html
Room 310, School of Mathematics
University of Birmingham
Friday 1st Nov 2013 -- 5pm

(DRAFT: Liable to change. Please save links, not copies.)

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This document is

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/mathsem.html>

It is part of the [Meta-Morphogenesis project](#).

A partial index of discussion notes is in

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html>

Further reading related to this project is [below](#) (to be extended).

Overview (Revised Dec 2013)

All human mathematicians are (in part) products of biological evolution. Why should members of any animal species have mathematical interests and capabilities? Because the world in which organisms develop, consume, compete, flourish and reproduce is rich in mathematical domains, and mathematical mastery of those domains brings biological benefits.

But that's not enough: the mechanisms of natural selection must have rich enough powers to generate mathematical mechanisms before mathematical mechanisms can be selected. So there may be universes, or planets, where the powers of natural selection are much more limited than they were on this planet 4.5 billion years ago. Those limited powers might not have supported evolution of the same range of mathematical opportunities, mechanisms, and powers as emerged on Earth. What made that possible? Answer: the original mechanisms and new ones produced by evolution? What original mechanisms? How were new ones produced?

At all stages there are **mathematical** structures involved in the evolutionary progress, and evolution "blindly" discovers and uses them.

In later stages of evolution, individual organisms develop abilities to discover and use mathematical structures and processes, though without realising what they are doing.

At that stage both evolution and some of its products including non-human species that are still flourishing, are "blind" mathematicians.

Later still, humans began to think explicitly about these processes and to discuss their properties. The earliest "non-blind" mathematicians used new biological mechanisms to notice and reflect on their mathematical reasoning.

Those developments led, eventually, to the collaborative production of Euclid's Elements. That was definitely a collection of mathematical discoveries even though neither the formalisms nor the proof methods used were of the type that many (though not all) mathematicians now seem to regard as defining mathematics, namely use of logically formulated axioms and rules of inference, and accepting only proofs that are logically valid derivations from axioms.

At present there's very little we know about the actual history of evolutionary developments and the pre-history of human mathematics. I'll try to show how the meta-morphogenesis (M-M) project sets out a strategy for trying to fill some of the gaps, with the hope of answering not only evolutionary questions, but also philosophical questions about the nature of mathematics and how mathematical reasoning and knowledge differs from other kinds. It also involves metaphysical claims about the possibility of finding mathematical structures in the world.

Perhaps the project will eventually help us to develop robots and AI systems that are far more intelligent than the current models are, and at last provide good explanatory models of human and animal intelligence.

This may also lead to much better mathematical education systems based on deeper insights into the nature of mathematics and into the nature of biological processes of learning and discovery.

NOTE:

Some of these ideas will be presented in a 'Frontiers of Science' talk at the Association for Science Education (ASE) conference in Birmingham, on 11th Jan 2014.

"Could a baby robot grow up to be a mathematician?"

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/ase-2014-abs.html>

Planes question (Part 1):

Is it possible for three plane surfaces to bound (completely enclose) a finite volume? Think about this and remember **how** you think about it. The question will be followed up below.

Comments, criticisms, suggestions, about the planes question or anything else here, to a.sloman@cs.bham.ac.uk

Introduction: Mathematics and Meta-Morphogenesis

"For mathematics is after all an anthropological phenomenon"

(L. Wittgenstein, *Remarks on the Foundations of Mathematics*)

I'll argue that Wittgenstein was completely wrong.

Mathematical activities include **biological** phenomena whose possibility depends on metaphysical facts but for which natural selection could not have achieved so much, starting from a lifeless planet.

The metaphysical facts are that there are infinitely many mathematical domains, including many whose instances are intimately involved in solutions to biological problems found by natural selection.

A mathematical domain is a (possibly infinite) collection of possible structures and processes. Many, though not all, mathematical domains can have physical instances, including physical and chemical structures and processes.

Many mathematical discoveries were made and used (blindly) by evolution and its products long before there were any humans on the planet. So anthropology, the study of **human** diversity, is too limited in scope to cover mathematics.

Many biologically important mathematical domains have instances that include construction and use of information structures. More examples are in [this draft list](#). Some of the links with biology are presented in outline below.

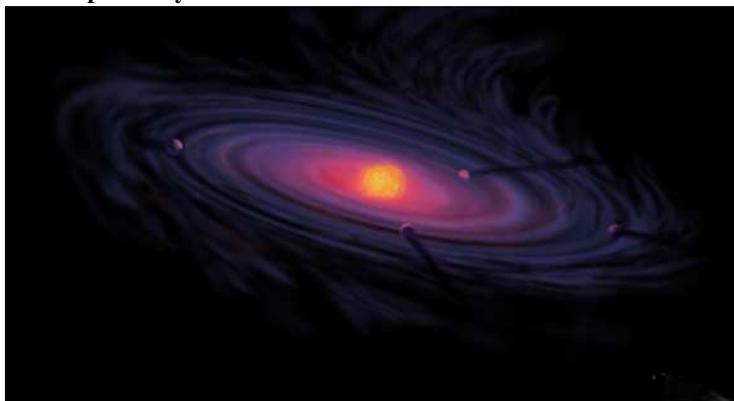
Other views of philosophers of mathematics are challenged by the biological role of mathematics, and the history of mathematical discoveries. One is the view of mathematics as essentially concerned with discovering logically derivable consequences of logically formulated sets of axioms. The question about planes, posed [above](#) and mentioned below illustrates this challenge to logicist, axiom-based theories about the nature of mathematics -- unless the reader is one of the few who use logic to reason geometrically.

What Alan Turing might have done.

One of Turing's last papers, on Chemical Morphogenesis (1952) -- now his most cited paper -- investigated ideas about how two chemicals diffusing at different rates, and interacting when they meet, could produce a wide variety of patterns on the surface of a developing organism, including dots, groups of dots, stripes, and blotches (as on a cow).

When I read it in 2011 in preparation for writing a paper about his work I wondered whether, if he had lived on, he might have combined this late work on mixed discrete and continuous control with his earlier work on discrete forms of computation, in Turing Machines.

A Protoplanetary Dust Cloud?



[NASA artist's impression of a protoplanetary disk, from Wikimedia]

Perhaps he could have made significant progress on a problem that has baffled biologists, chemists, physicists and others: how a planet-wide ecosystem full of billions of living things of millions of types [*], including mathematicians, could have assembled itself from a state with no life, but with some external sources of energy, including solar and cosmic radiation and asteroid impacts.

[*] <http://www.bbc.co.uk/news/science-environment-14616161>

BBC News, 23 August 2011: Species count put at 8.7 million

I can't do what Turing would have done, but I've given a Turing-inspired name to the project, namely "Meta-Morphogenesis" (since products of morphogenesis in evolution change the mechanisms of morphogenesis), and I've begun to identify some of the *types* of work to be done, including a major multidisciplinary collaborative investigation of changes in forms of **information-processing**.

That investigation complements existing biological studies of changes and variations in morphology, in behaviours, in habitats, in modes of reproduction, and in DNA.

This investigation includes assembling and organising evidence-based, theory-based and conjectural examples of such transitions. Then, as evidence accumulates and our theories grow, we may think of new explanations of how these changes came about, and their implications for later changes. This task will be easier if guided by ideas about many intermediate stages in the transitions from molecules to mathematicians, some of which are conjectured below. A longer list, addressing a wider range of evolutionary transitions can be found [here](#).

A (messy) overview of the whole M-M project can be found on this web site:

<http://tinyurl.com/CogMisc/meta-morphogenesis.html>

Why we need the M-M project

Psychologists, neuroscientists, cognitive scientists, AI researchers (including roboticists) and others, already attempt to find out what evolution produced and how the products work, by studying existing systems and trying to unravel and model their mechanisms. I suspect these approaches cannot succeed, though they produce many indications of progress -- but progress towards partial answers only.

One of the problems of studying current species, especially recently evolved species, is the invisibility of most of the information processing and the unobviousness of many of the requirements that drove evolution of the functions and mechanisms. A related problem is the rich interconnectedness of solutions to various sub-problems. Attempts by neuroscientists and psychologists to do laboratory experiments probing separate mechanisms still depend on the whole human to understand instructions and perform actions. So what appear to be properties of sub-mechanisms may merely be properties of experimental paradigms.

If we can use the observed existing diversity of competences, fossil records, and many other clues, to suggest (a) intermediate forms of information processing and how they changed, and (b) possible mechanisms supporting those forms of processing, then we may be able to build a very large map that will continually guide further research, both towards new contents for the map, and also new explanatory mechanisms and architectures. Merlin Donald ([2002](#)) also uses evolutionary considerations to drive informed speculation about the architecture of a human mind, but his ideas about information-processing mechanisms do not seem to be based on practical model building experience.

Aspects of evolution:

Generative mechanisms and selection mechanisms

The explanatory power of natural selection includes one element that is often cited and one that is rarely cited -- at least in popular presentations. The first is the ability of some biologically useful changes to be preserved and the numbers of individuals inheriting some of those changes to grow faster than numbers of individuals without those changes, in a competitive environment.

The second, more fundamental, pre-requisite for natural selection is the existence of underlying **generative mechanisms** of reproduction and development that are capable of producing novel structures, mechanisms, forms of information-processing, and behaviours, from which natural selection can choose useful subsets.

Not all physical substrates would be capable of supporting mechanisms with those properties. A planet formed only from grains of sand held together by gravitational forces would probably not support such generative mechanisms, and in that case evolution of the biological diversity found on our planet could not have happened. There may be different collections of generative physical mechanisms that all support some sort of evolutionary change. But only a special subset could generate the opportunities for selection that occurred on this planet -- and they depend crucially on the properties of chemistry, as contrasted, for example, with the properties of point masses and rigid bodies studied in Newtonian mechanics. (Some of the reasons for the importance of chemistry are presented in the work of Tibor Ganti (2003) and Stuart Kauffman).

If we can understand the requirements for those mechanisms, and how chemical mechanisms, with their rich mixture of continuous and discrete processes, meet the requirements for information-processing in living things, and why Turing machines and logical or arithmetical inference engines do not, we may be better able to understand how we, and many other life forms, evolved. In particular, that might explain the evolution of mechanisms supporting mental states and processes, including mechanisms supporting processes of mathematical discovery.

(See the Further Reading below.)

Of course, this challenges the hypothesis that a Turing machine, or collection of Turing machines, could simulate everything in the physical universe.

The mixture of physical and chemical structures and processes produced when this planet formed had the potential to support a particularly rich variety of possible structures and processes of varying size and complexity.

It also (crucially) had the ability to support more and more diversity as size and complexity increased, including discontinuous changes in types of information processing -- e.g. from handling information about current sensory and motor signals to handling information about things that endure when not sensed or acted on (from a somatic ontology to an exo-somatic ontology).

It seems that evolution had to be able to produce some major discontinuities in functionality and design. Those transitions include transitions in **mathematical domains** instantiated in products of evolution.

Example:

Continuous change in a plane surface can produce a curved line growing longer and longer, with a fixed curvature. But when when the "growing" end meets the other end there is a major discontinuity, partitioning the whole plane into two disconnected regions. Such discontinuities can occur in evolution and in development.

(Is Catastrophe Theory relevant? http://en.wikipedia.org/wiki/Catastrophe_theory)

I suspect that many different mathematical domains have been directly involved in providing new opportunities throughout evolution, with the variety of domains and options increasing as complexity of organisms increased.

Mathematics can not only generate ever increasing complexity and diversity of structures and processes, it also provides opportunities to tame complexity and diversity through the discovery of powerful new abstractions that can be instantiated

in diverse ways, a point made in this document on The Nature of Mathematics, although I think it over-emphasises the role of logic in mathematics:

<http://www.project2061.org/publications/sfaa/online/chap2.htm>

American Association for the Advancement of Science, 1990.

I suspect that long before human mathematicians discovered new abstractions to tame complexity, natural selection did that, many times, as explained below.

Mathematical domains required for evolution to work (Expanded: 21 Nov 2013)

A domain is a set of related possible structures and processes. Processes involve transformations between structures. The processes can include addition or removal of sub-structures (or both) and alterations of properties and relationships. Any actual situation will typically involve instances of several domains, along with possibilities for change in accordance with processes in those domains.

For example, a sphere located on a surface is an instance of a domain that includes possibilities of sliding, rolling along the surface and rotating at a point on the surface, all of which are processes that can introduce new relationships. If the sphere has marks on the surface those the processes will also alter relationships between those marks and parts of the plane, or other things on the plane. If there are other objects on the plane, then possible motions of the other objects also constitute domains. Domains can be combined to form larger domains, or subdivided into smaller domains, as implied in the description above of different possible forms of motion of the sphere on the plane.

One of the sources of mathematics is the existence of constraints in many domains.

For example among the domain of straight line segments in a plane, one of the constraints, that may not be obvious to everyone at first, is that no two straight lines can enclose a bounded region of the plane. Why not? Think about possible configurations of two straight line segments and how they can separate parts of a plane from other parts.

Atoms and molecules and their possible interactions provide an enormous variety of domains of structures and processes, which can be combined in various ways to produce ever more complex domains. Among the (relatively late) products of biological evolution are organisms that can construct and think about domains of which perceived physical configurations are instances. J.J.Gibson's notion of perceiving affordances includes a small subset of such cases. Different organisms can detect and reason about different sets of affordances -- in different domains. But evolution itself detects uses domains in developing its abilities to produce varying types of organisms with varying sorts of capabilities.

Living organisms need many types of parts, made of many types of materials, with many types of relationships, forming many types of structure, and interacting in many types of ways to produce many types of process that produce and change many types of state. Many types of material, relationship, structure, interaction, process, and state have properties that can be characterised mathematically, and can be broken down into instances of different mathematical domains: domains concerned with spatial structures, spatial processes, forces, energy transfer, and also types of information, forms of representation, forms of information-processing, including especially types of **control** -- perhaps the most basic use of information, from which others (such as referring, stating, asking, intending) can be constructed.

(Sloman 1985), (Sloman 2011)

The more complex structures, processes and capabilities cannot (normally?) be assembled by gradual accretion of the simplest elements. Rather, achieving a particular level and type of complexity, with new building blocks and relationships for assembling novel structures, may be required for some further increases in biological sophistication to become possible. The evolutionary transition from motion on four limbs to bipedal walking and running could not have passed through all possible waist angles if forelimb lengths did not increase enormously. When appropriate geometry and muscle strengths had developed, perhaps some discontinuous change in control mechanisms in the brain then enabled upright walking and running. If so, evolution of locomotion may have passed through many slightly different mathematical domains and also made some jumps between very different domains. Similar discontinuities have occurred in the history of computing systems design in the last 70 years or so. We now have many kinds of virtual machine that can provide *platforms* on which more sophisticated types can be built but which could not be built simply by specifying manipulations of bit patterns in the computer's memory.

[To be added: Why not?]

Note (Added 21 Nov 2013):

A large topic that will be discussed separately later is the difference between the idea of a set of possible worlds, and a domain. I have many doubts about the usefulness of the notion of 'this world' let alone reference to sets of possible alternatives to this world. Moreover it is clear that there are many people who are able to think about possibilities and constraints on possibilities without thinking about alternative possible worlds. Instead they think about alternative configurations of a portion of the world, and can discover constraints limiting such alternatives. So I suggest that our normal "modal" concepts of possibility and necessary derive from an understanding of domains and invariants in domains, which in turn derives from earlier biological mechanisms for handling sets of possibilities in some portion of the world.

Even if it is possible to give a coherent analysis of what "this world" is and what the alternative possible worlds are, that would turn out merely to be a special, very complicated case, of this more primitive notion of a domain including many possibilities. For a partial discussion of this, notion, and its importance for intelligent agents, see [Sloman 1996](#).

The relevance of pictures of impossible objects to understanding human and animal perception of space is discussed (briefly) in a paper on unsolved problems concerning animal and robot vision in this section, and the preceding section on partial orders:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/vision/#impossible>

Conjecture: "Blind" mathematical discovery

Evolution initially "blindly" discovered and made use of many kinds of mathematics, insofar as many products of natural selection depend on the existence of classes of structures and processes with mathematical properties that strongly constrain the structures and processes that their interactions can produce.

A simple example is the need for some rigid materials and some flexible materials. Rigidity and flexibility are properties of materials that constrain the properties of processes in which they are involved: both sorts can change their locations relative to other objects, whereas rigid materials prevent changes of spatial relationships

between parts of an object and flexible materials do not. Flexible materials can have different sorts of mathematical properties affecting the types of change that can occur, e.g. elastic, and inelastic changes. An extreme case of flexibility is being a liquid.

A more complex example is the topological property of hollow 3-D impermeable (or, more importantly, semi-permeable) membranes: separation of different mixtures of chemicals inside and outside the membrane, thereby maintaining an internal combination that would otherwise be diluted and polluted. Mechanisms for forming such structures and allowing them to reproduce with their contents seem to have been early products of evolution. Tibor Ganti (2003).

A related example is use of selective, controlled, flow through such an enclosing membrane to allow useful supplies of materials and energy to come in and waste products to go out. Kinematic topology?

Another example is use of sensor values to modify mechanisms that allow rates of flow to increase or decrease: homeostatic mechanisms whose mathematical properties prevent fatal monotonic or violently fluctuating changes.

Over millions, or billions, of evolutionary steps there were many other mathematical discoveries, some concerned with properties of structures, e.g. how to make strong, light skeletons, how to give joints or contact points the right sort of compliance to support accurate control across a range of task situations, how to configure joints and bone structures to allow required patterns of movement at the far ends of limbs, how to produce an increasing variety of external and internal sensors capable of acquiring information of many kinds, how to use information in increasingly diverse ways including immediate use control of behaviour and postponed use when the opportunity arises (e.g. retaining information about location of a food store found while foraging).

Later, as the information to be acquired, manipulated, derived, stored, combined, and used became increasingly complex (e.g. information about entities enduring when unobserved with some changing and some unchanging properties and relationships, including enduring spatial relationships in some cases), evolution discovered forms of information acquisition, storage and manipulation that allowed such information to be used at different times, e.g. in planning manipulative actions on objects, or planning routes across extended terrain.

In doing all that it implicitly **proved** that various design solutions were possible, by creating working instances. In short, the "blind watchmaker" metaphor for natural selection may be less illuminating than a "blind theorem-prover" metaphor. The corresponding proofs would be the evolutionary steps from some early state of the planet to the existence of working instances of new structures or mechanisms.

Alas, the vast majority of such proofs have not been recorded, and reconstructing them will be very difficult, using much guess-work and conjectural interpretation of evidence. But if enough linked chains of hypothesised proof-steps can be linked to form a coherent large network, we one day have the best evidence that can be hoped for, even if it is not conclusive.

Getting individuals to take on the burden of discovery

Making all those discoveries by natural selection is abominably slow, but evidently evolution found ways of speeding up its operation, by making mathematical discoveries about classes of structures, processes, and information, and producing species whose individuals could instantiate those abstractions differently in different environments, by interacting with the environments, thereby shifting part of the discovery process from

evolution to processes of development and learning, which are much faster.

This is related to what has been called "The Baldwin Effect" briefly discussed in http://en.wikipedia.org/wiki/Baldwin_effect

Some of the discovery processes were collaborative: individuals moving around responding to their immediate environment and current needs could leave chemical trails and the combinations of chemical trails of many individuals could change the environment into an important source of information about where to go. This is one of many examples of "stigmergy" <http://en.wikipedia.org/wiki/Stigmergy>

It seems that later on some of those processes were mirrored in or replaced by changes in brain structures in individuals -- e.g. since stable pheromone trails are not usable by flying animals or animals in fast flowing water.

Note: Some changes to the environment produced by collective behaviour of individuals last a short time then are removed by influences such as rain, wind, sandstorms, etc. Others, such as paths hacked through forests where trees do not reappear quickly, roads built with enduring materials, tunnels and pyramids can endure long enough to affect many generations. I suspect there are many examples of environmental changes produced by a species having a long term effect on the genome, in the same way as climatic or other externally produced changes can.

An example may be the evolution of abilities to use languages for communication, after evolution of "internal languages" required for perception, control of actions, learning, motivation, intention, plan formation,, intelligent collaboration, imitation, etc. Some speculations about how that might have happened can be found in <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#glang> (Compare Dawkins on "The Extended Phenotype").

The evolved information storage mechanisms include (at least) two mathematical domains:

(a) the domain of information **contents**, e.g. facts about terrain, and (b) the domain of structures and processes in the information **medium** (chemical trails, neural connections, stored data-structures, etc.)

It is often assumed, wrongly, that isomorphism between related components of type (a) -- semantic contents -- and type (b) -- syntactic structure -- is necessary.

Getting the right mathematical relationships between different parts of the genome controlling different aspects of physical, behavioural, and information processing capabilities is extremely difficult. Perhaps evolution found ways of coordinating some of the changes it produced, e.g. relating the internal information structures for controlling actions, to the degrees of freedom and scale of movement of the effectors.

This makes evolution a blind theorem prover -- where the theorems are about what works under what conditions, and the derivations are the evolutionary trails. At that stage there may be nothing in biology that can distinguish pseudo-theorems that just happen to be useful in a combination of circumstances that cannot be relied on to persist, and theorems that cannot possibly have exceptions because of the mathematical relationships. Discovering a pattern of motion that permits escape from a predator with particular appearance might be an example of the former. Discovering a way to modify the second derivative of a homeostatic loop as well as the first derivative in order to prevent overshooting would be an example of the second.

My observations suggest that human children spend far more time exploring different mathematical domains in the first few years of life than anyone has noticed. This is driven by what I've called architecture-based motivation rather than reward-based motivation, since the individuals have no possible way of knowing what rewards they will get from their explorations, and with architecturally triggered motives they do

not need any rewards as explained in this paper:

<http://www.cs.bham.ac.uk/research/projects/cogaff/architecture-based-motivation.pdf>

A video recording of an 11 month old child apparently exploring several different domains associated with a piano can be viewed as video 4 [here](#). The other videos on that page illustrate other examples of exploration of structured domains.

It seems that Emre Ugur has demonstrated such processes of active exploration, followed by re-organisation of information, in a robot, in his [PhD Thesis \(2010\)](#) and related publications.

Providing mathematical meta-cognition for some species

Later still, evolution may have found ways to produce organisms that, in addition to having the above abilities, also have an extra (meta-cognitive) layer of processing that develops relatively late and inspects what has been learnt so far, and then finds ways of improving it, and perhaps later on, or in parallel, develops mechanisms that allow individuals making such discoveries to communicate the results to others, permitting cultural evolution to speed up natural selection. Later still evolution might extend the mechanisms to allow not only results but improved methods of discovery to be noticed, thought about, and communicated, and later jointly debugged. Compare Karmiloff-Smith, 1992. <http://tinyurl.com/CogMisc/beyond-modularity.html> I suspect the more intelligent of current biological species depend on many different examples of earlier transitions in types of metacognition, in different environments, on different scales, with widely varying mathematical complexity and power. It may take many decades, or perhaps centuries, of interdisciplinary collaboration to fill in the gaps and extend our knowledge to the level that will permit us to build machines with squirrel-like or crow-like, or elephant-like or ape-like or human-like capabilities.

Collaborative mathematical meta-cognition

They will need the ability to make mathematical discoveries and share and criticise them in collaborative ways, perhaps in something like the processes by which our ancestors made discoveries that contributed to Euclid's Elements. I suspect some of those abilities will fit Immanuel Kant's specification for the ability to discover synthetic necessary truths (in his **Critique of Pure Reason**):

<http://www.marxists.org/reference/subject/ethics/kant/reason/ch01.htm>

"THERE can be no doubt that all our knowledge begins with experience. For how should our faculty of knowledge be awakened into action did not objects affecting our senses partly of themselves produce representations, partly arouse the activity of our understanding to compare these representations, and, by combining or separating them, work up the raw material of the sensible impressions into that knowledge of objects which is entitled experience? In the order of time, therefore, we have no knowledge antecedent to experience, and with experience all our knowledge begins.

But though all our knowledge begins with experience, it does not follow that it all arises out of experience. For it may well be that even our empirical knowledge is made up of what we receive through impressions and of what our own faculty of knowledge (sensible impressions serving merely as the occasion) supplies from itself. If our faculty of knowledge makes any such addition, it may be that we are not in a position to distinguish it from the raw material, until with long practice of attention we have become skilled in separating it. This, then, is a question which at least calls for closer examination, and does not allow of any off-hand answer: -- whether there is any

knowledge that is thus independent of experience and even of all impressions of the senses. Such knowledge is entitled a priori, and distinguished from the empirical, which has its sources a posteriori, that is, in experience. ...

...

In what follows, therefore, we shall understand by a priori knowledge, not knowledge independent of this or that experience, but knowledge absolutely independent of all experience. Opposed to it is empirical knowledge, which is knowledge possible only a posteriori, that is, through experience.

...

We should soon be able to say more than Kant, e.g. about the meta-competences, meta-meta-competences, meta-meta-....competences provided by evolution and by genomes interacting with individual development, that make all this possible.

For an introduction to a distinction between pre-configured and meta-configured competences (Jackie Chappell and Aaron Sloman, 2007) see:

<http://www.cs.bham.ac.uk/research/projects/cosy/papers/#tr0609>

To be continued, with more examples of mathematical competences that normally go unnoticed.

For examples of "toddler theorems" see:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html>

Planes question (Part 2):

The question was:

Is it possible for three plane surfaces to bound a finite volume?

If you have answered it can you specify how you answered it?

Did you derive the result from a set of axioms?

If you have not found an answer to the question, consider this alternative question:

Is it possible for two straight lines to bound (completely enclose) a finite area?

If not, why not?

If it is possible, then please provide an example (and email me!).

The questions about what can be enclosed by planes and lines are concerned with spaces in which entities can move and change relationships continuously. Not all mathematical domains are like Euclidean space.

There are some hybrid domains that are partly like euclidean space and partly discrete. The next section provides examples that not everyone will find familiar.

Example: diagonal motion on a rectangular grid:

Suppose you have a large rectangular grid made of small squares arranged in lines and columns. The grid can be arbitrarily large.

Case 1:

Suppose you have four blue buttons and four red buttons, in arbitrary locations on the grid. You are allowed to move the blue buttons but only diagonally (through the corners of the squares, not horizontally or vertically).

Case 2:

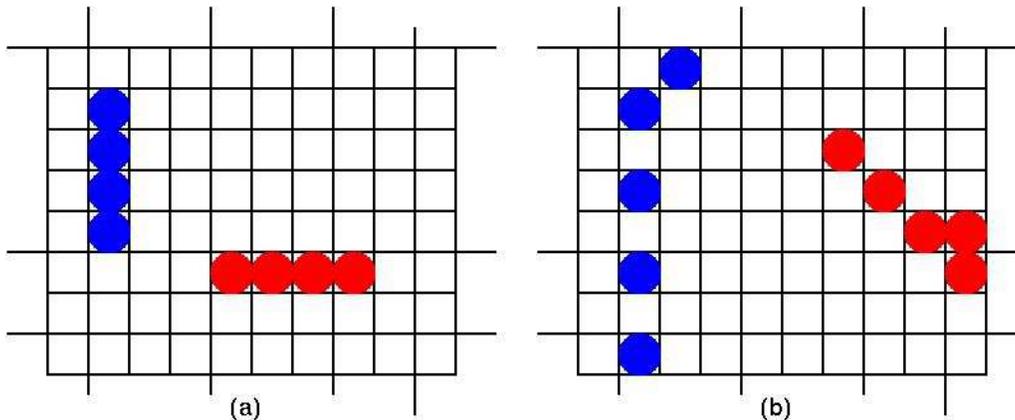
Like case 1, but with five red and five blue buttons.

Question:

Using only diagonal moves of the blue buttons

will it always be possible to get each blue button to a red button, so that each red button has a different blue button on it?

- Does the answer depend only on the number of buttons of each colour (as long as they are the same)?
- Does it depend on the size of the grid?
- Does it depend on where the buttons are located?



Can the blue buttons in grid (a) be transferred to the locations of the red buttons using only diagonal moves. What about grid (b)?

Supplementary question:

Does the answer depend on whether the red and the blue buttons are arranged in similar ways, e.g. both in a horizontal row, or both in a vertical row, or one vertical one horizontal, or some other configuration?

Does the answer depend on the fact that five is a prime number and four is not?

Does extending the grid so that larger detours are possible make a difference to the answer?

If you find **both** an answer to a question **and** a proof that it is a correct answer I'll be interested to learn how you proved it. There isn't one **right** way, though some proofs are more elegant and more general than others. Can you characterise the domain of structures (including processes) to which your proof applies?

An alternative exercise

In how many different ways can you prove that interior angles of a planar triangle add up to half a rotation (=180 degrees)?

What happens to your proof on the surface of a sphere.

To what mathematical domain of structures is your proof relevant?

To find out more about that problem see

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html>

Question

What are the information-processing requirements for a future robot to be able to think about such problems and discover proofs?

Can computer-based machines do this now? Discussed in:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html>

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html>

See also Mary Pardoe's proof of the triangle sum theorem:

NOTE (Added 5 Nov 2013):

Lurking behind the scenes in this document is my belief that Immanuel Kant's theory that mathematical discoveries lead to what he described as "synthetic a priori knowledge" is basically correct, even though he used Euclidean geometry as an example and Euclidean geometry turned out to have an empirical component that was refuted by Eddington's observation of stellar shift during a solar eclipse in 1919, which confirmed Einstein's prediction based in the general theory of relativity, according to which space is non-euclidean in the presence of physical masses. The argument (which I encountered among philosophers in Oxford in the late 1950s), that Kant had been shown to be wrong by Einstein ignored the richness of Euclidean geometry and the fact that the vast majority of it remained intact. After I learnt about Artificial Intelligence and started programming, around 1970 I acquired the ambition to show that Kant was correct, by demonstrating how his claimed processes of non-empirical discovery, triggered by empirical discoveries, but based on genetically endowed abilities to find a deeper structure, could be demonstrated in a 'baby' robot interacting with its environment and making discoveries in something like the way human children do, including discoveries of the sort that might have contributed to Euclid's elements, and other forms of mathematics.

For various reasons this challenge turned out to be far more difficult than I expected, so the ideas presented here are merely a report on work in progress, in a difficult long term research project.

Planes question (Part 3):

(Updated: 22 Nov 2013)

The question was:

Is it possible for three plane surfaces to bound a finite volume?

The alternative, easier, question was:

Is it possible for two straight lines to bound (completely enclose) a finite area?

I believe it is possible for non-mathematicians to think about both questions, especially the second, and reach a conclusion that is essentially understood as a necessary truth (e.g. two straight lines cannot bound a finite area, though they can (if they are infinitely long) divide the plane containing them into three or four disjoint infinite areas).

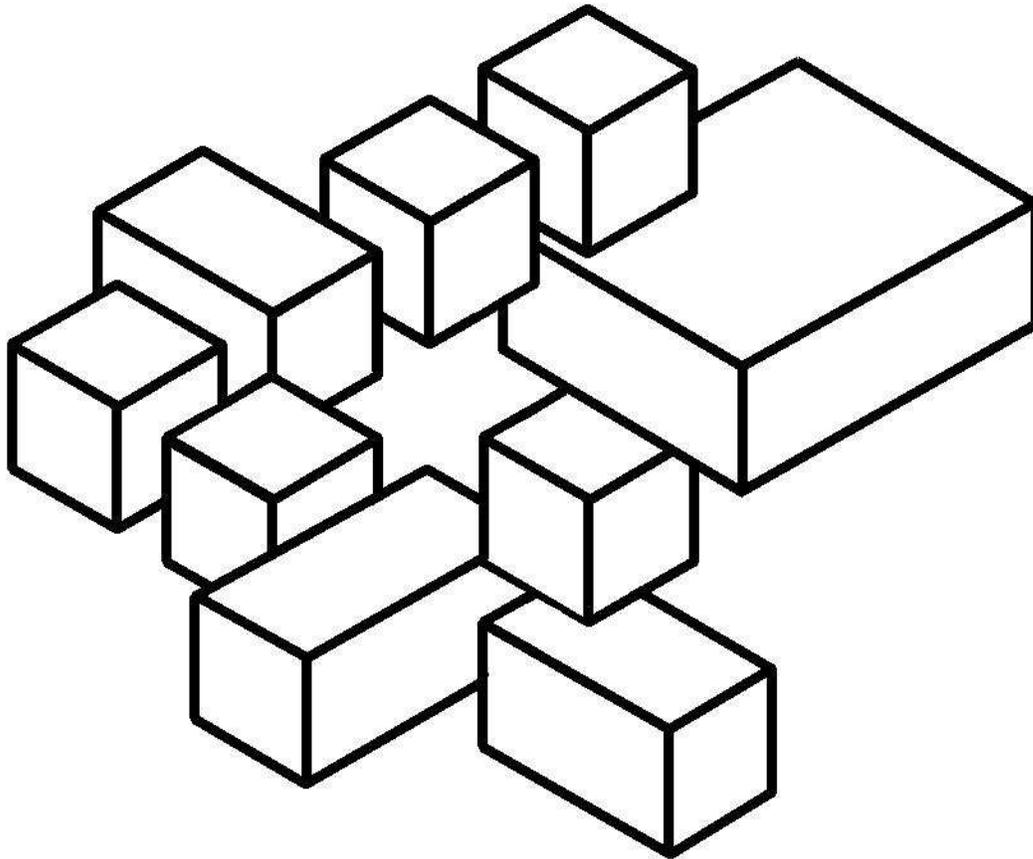
Likewise by considering alternative possible configurations of flat surfaces, first one surface, then two, then three, you should be able to convince yourself that three planes cannot completely enclose a volume though they can divide a 3-D space into separate volumes in various ways (one way would be as an infinite tube of fixed cross-section).

Sometimes impossibilities are unobvious. For example the painting by Pieter Bruegel entitled "The Magpie on the Gallows" (1568) includes a 3-D object that is impossible if all the parts of the gallows are interpreted in a natural way, though the impossibility may not be noticed at first: <http://im-possible.info/english/art/classic/bruegel.html>

Bruegel was not the only artist to discover so-called "impossible objects" before Roger Penrose discovered his impossible triangle. The Swedish artist Oscar Reutersvärd produced a similar but more interesting version in 1934.

If you were provided with a pile of rectangular blocks of various sorts could you

construct a configuration like the one depicted in the picture below?



That example was inspired by the first of Reutersvärd's impossible objects, apparently discovered while developing a doodle derived from a star, shown on the commemorative stamp [here](#).

The main point I want to make is that many humans, presumably including the original developers of Euclidean geometry are able to think about a type of spatial configuration, specified either verbally or in a picture of how it should look, or a specification of how to construct one, and then discover that the configuration is impossible, even if many that are close to it are possible. Adding the last brick will always produce something different from what is depicted or otherwise specified. The discovery of such impossibilities seems to make use of deep biological capabilities required for reasoning about positive and negative affordances in the physical environment, or more generally reasoning about possible ways in which the environment might have been different or could be made different from how it is. Right now I don't think there are any AI reasoning systems, nor any known brain mechanisms, capable of explaining how such discoveries are made. One thing is clear: we do not need to start from a set of axioms expressed in a logical formalism and then derive a formula stating the impossibility. The possibility of constructing that sort of proof was discovered long after Euclid, and it should be obvious that whatever formalisation of geometry is used, the discovery that formula F ('such and such is impossible') can be derived logically from a set of axioms is not the sort of discovery that I have been talking about and which led to Euclid's elements. Rather, possibilities and impossibilities (including positive and negative affordances) can be discovered by inspecting particular configurations in a certain way that I

suspect has never been characterised accurately, though Kant seems to have been the first to try, and many anti-logicist, anti-formalist mathematicians have tried since Kant. For more on this see the section on impossible objects in the discussion of gaps in current theories and models of visual perception: [here](#), and the discussion of "hidden depths in triangle qualia" [here](#).

Conclusion (provisional)

Since the precursors of Euclid whose discoveries led to Euclid's Elements, and the branch of mathematics that we now think of as Euclidean geometry, could not have used explicit axioms and explicit inference rules in making those discoveries, there must be a kind of mathematical discovery procedure that is independent of such apparatus. I believe that is close to what Immanuel Kant was claiming when he argued that mathematical knowledge could be necessary (non-contingent), non-empirical (though "awakened" or "triggered" by experience) and synthetic (not true by definition). The later development of the axiomatic method merely introduced a new branch of mathematics, and confused some mathematicians (not all) into thinking that was the only true mathematics. Benoit Mandelbrot fulminated against that view of mathematics at guest lecture at IBM UK about 20 years ago.

I agree with him. The question about planes bounding a volume was discussed in my DPhil thesis 1962, as one of many examples supporting Kant.

<http://www.cs.bham.ac.uk/research/projects/cogaff/07.html#706>

The questions about sliding blue and red buttons around a rectangular grid can easily be answered if you notice that the grid squares can be divided into two colours, e.g. white and black on a chess board, and consider what changes diagonal moves of a button can make.

Although the plane surface is continuous, and **motions** of buttons are continuous, in the domain of motions between squares there are discontinuities in **relationships** that turn this into a partly discrete domain.

I have presented the questions about what planes and straight lines are capable of enclosing and the questions about sliding buttons, because I think everyone able to read this document is also capable of making mathematical discoveries, in something like the way precursors of Euclid might have done (possibly with a little help in some cases). If you do that you will have the type of experience that I think Kant was attempting to characterise in his theory about the nature of mathematical discovery.

In doing so, you do not need to start from a set of axioms specifying assumptions in a logical notation, nor a set of logical inference rules against which you check your reasoning steps. That sort of activity defines a type of mathematical domain that was not discovered by mathematicians/logicians until the nineteenth century, although Aristotle, Leibniz and probably others took some preliminary steps.

We shall have a better understanding of the nature of mathematics when we know how to build artificial mathematicians that can make discoveries in the same sorts of ways as human mathematicians have been doing for over two thousand years.

So far we don't know how to do that, despite some partly relevant artificial reasoners, that can go through the motions, though without knowing what they are doing or why, in most cases.

Note that I am not claiming, and I don't believe Kant was claiming, that humans are infallible when they discover proofs. It is clear from the work of [Imre Lakatos](#) that mathematicians, even great mathematicians and communities of mathematicians, can

make mistakes of various kinds, e.g. failing to notice special cases. However, mathematical mistakes of various kinds can be detected and corrected: and that is part of the normal process of mathematical discovery. I have some examples in a discussion of ways of thinking about areas of triangles here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html>

I'll add more on that later.

NOTE: I am not claiming that I have explained how the forms of mathematical reasoning discussed here are possible. One form of explanation would be a specification for design of a robot that can be shown to enable a 'baby' robot to make mathematical discoveries as humans do partly as a result of exploring the environment and partly as a result of use of very powerful forms of reasoning whose possibility is somehow encoded in its "genome", but which has to be developed by being applied to the environment.

Compare John McCarthy on [The Well Designed Child](#).

and Chappell and Sloman (2007) on pre-configured and meta-configured competences:

<http://www.cs.bham.ac.uk/research/projects/cosy/papers/#tr0609>

Further Reading

(To be expanded)

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A DRAFT list of types of transition in biological information-processing
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Meta-Morphogenesis and Toddler Theorems: Case Studies
- <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/bio-math-phil.html>
Biology, Mathematics, Philosophy, and Evolution of Information Processing
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#glang>
Talk 52: Evolution of minds and languages.
What evolved first and develops first in children: Languages for communicating, or
languages for thinking (Generalised Languages: GLs)
- <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/vm-functionalism.html>
Virtual Machine Functionalism
(The only form of functionalism worth taking seriously in Philosophy of Mind)
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TO BE EXPANDED

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