# Parametricity and excluded middle

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**Abstract**

In univalent foundations, it is known that the law of excluded middle allows one to define a family of functions $f_X : X \to X$ that is not the identity function on the booleans. We show that the converse holds as well: given such a function, we derive the law of excluded middle.

Suppose we are given a polymorphic function

$$f_X : X \to X,$$

where $X : \mathcal{U}$ is its type parameter.

If this were a term in a language such as System F, then parametricity tells us that it must be equal to the identity function $\mathsf{id}_X$ for every type $X$. But parametricity is a metatheoretical framework: it gives properties about the terms of a language, rather than internally stating properties of elements.

Internal to univalent foundations, if we have LEM, then there exists a polymorphic function $f$ such that $f_{\mathbf{2}}$ (where $\mathbf{2}$ is the type of booleans) is not the identity function [2, exercise 6.9]. Since LEM is consistent with univalent foundations, this means that there cannot be an internal proof that a polymorphic function $f_X : X \to X$ is equal to the identity.

We prove that, in univalent foundations, LEM is precisely what is needed to get a function family not equal to the identity on $\mathbf{2}$: on the one hand, we already know that LEM gives us such a function; on the other hand, we have the following converse.

**Theorem 1.** *If there is a function $f : \Pi_{X:\mathcal{U}} X \to X$ with $f_{\mathbf{2}} \neq \mathsf{id}_{\mathbf{2}}$, then LEM holds.*

Alternatively, to confine the amount of univalence needed, we can work in the setting of intensional type theory with function extensionality (but without full univalence), and assume that $f$ is *extensional* in the sense that it is invariant under equivalences on the type $X$ it acts on.

The idea of the proof is that we define a type $\mathbf{3}_P$, which, depending on whether $P$ holds, may or may not be equivalent to $\mathbf{2}$. We then evaluate $f$ at the type $\mathbf{3}_P \simeq \mathbf{3}_P$ (rather than $\mathbf{3}_P$ itself), and prove $P + \neg P$ using that evaluation.

*Proof.* Without loss of generality, we may assume that $f_{\mathbf{2}}(0_{\mathbf{2}}) \neq 0_{\mathbf{2}}$.

To prove LEM, let $P$ be an arbitrary proposition. We need to prove $P + \neg P$.

We will consider a type with three points, where we identify two points depending on whether $P$ holds. Formally, this is the quotient of a three-element type, where the relation between two of those points is the proposition $P$. This quotient can be defined as

$$\mathbf{3}_P :\equiv \Sigma P + \mathbf{1},$$

where $\Sigma P$ is the *suspension* of $P$[1]. The two points of the suspension are called $\mathsf{N}$ and $\mathsf{S}$, and the identity path (if it exists) between those points is called $\mathsf{merid}(p) : \mathsf{N} = \mathsf{S}$, with $p : P$.

Recall the following about suspensions.

- By induction, we can define a map

  $$\mathsf{swap} : \Sigma P \to \Sigma P$$

  that sends $\mathsf{N}$ to $\mathsf{S}$ and vice versa.

- By induction, we can define a map $\mathsf{extract} : \mathsf{N} =_{\Sigma P} \mathsf{S} \to P$, and this can be generalized to a map

  $$\mathsf{extract}'_x : x =_{\Sigma P} \mathsf{swap}(x) \to P.$$

Notice that if we have $P$, then the suspension is contractible, so $\mathbf{3}_P \simeq \mathbf{2}$, and also $(\mathbf{3}_P \simeq \mathbf{3}_P) \simeq \mathbf{2}$.

Define

$$g :\equiv f_{\mathbf{3}_P \simeq \mathbf{3}_P}(\mathsf{ide}_{\mathbf{3}_P}) : \mathbf{3}_P \simeq \mathbf{3}_P,$$

where $\mathsf{ide}_{\mathbf{3}_P}$ is the equivalence $\mathbf{3}_P \simeq \mathbf{3}_P$ given by the identity function on $\mathbf{3}_P$. We will see $g$ both as an equivalence and as a function $\mathbf{3}_P \to \mathbf{3}_P$.

Now we do case analysis on $g(\mathsf{inr}(\star))$. Notice that this case analysis is simply an instance of the induction principle for sum types. In particular, we do not require decidable equality of $\mathbf{3}_P$ (which would already give us $P + \neg P$, which is exactly what we are trying to prove). When analyzing the case $\mathsf{inr}(t) : \mathbf{3}_P$, with $t : \mathbf{1}$, we are free to specialize to $t = \star$ since $\mathbf{1}$ is contractible.

**$g(\mathsf{inr}(\star)) = \mathsf{inr}(\star)$:** Assume that $P$ holds. Then by transporting the witness of $f_{\mathbf{2}}(0_{\mathbf{2}}) \neq 0_{\mathbf{2}}$ along an equivalence that identifies $0_{\mathbf{2}}$ with $\mathsf{ide}_{\mathbf{3}_P}$, we get that $g \neq \mathsf{ide}_{\mathbf{3}_P}$. However, since $\mathbf{3}_P \simeq \mathbf{2}$ and $g$ has a fixed point $\mathsf{inr}(\star)$, we can deduce that $g = \mathsf{ide}_{\mathbf{3}_P}$, which is a contradiction.

[1] The suspension of a type is not generally a quotient, because it is not generally a set: we use the fact that $P$ is a proposition here.

$g(\mathsf{inr}(\star)) = \mathsf{inl}(x)$**:** We do further case analysis on $g(\mathsf{inl}(x))$.

$g(\mathsf{inl}(x)) = \mathsf{inr}(\star)$**:** We do further case analysis on $g(\mathsf{inl}(\mathsf{swap}(x)))$.

$g(\mathsf{inl}(\mathsf{swap}(x))) = \mathsf{inr}(\star)$**:** Since we now have

$$g(\mathsf{inl}(x)) = \mathsf{inr}(\star) = g(\mathsf{inl}(\mathsf{swap}(x)))$$

and since $g$ is an equivalence, we can use $\mathsf{extract}'_x$ to get $P$.

$g(\mathsf{inl}(\mathsf{swap}(x))) = \mathsf{inl}(y)$**:** Assume $P$, in which case $x = \mathsf{swap}(x)$. Hence $\mathsf{inr}(\star) = \mathsf{inl}(y)$ which is a contradiction.

$g(\mathsf{inl}(x)) = \mathsf{inl}(y)$**:** Assume $P$, in which case $\mathsf{inl}(x) = \mathsf{inl}(y)$. But we now have

$$g(\mathsf{inr}(\star)) = \mathsf{inl}(x) = \mathsf{inl}(y) = g(\mathsf{inl}(x)).$$

So since $g$ is an equivalence, this yields $\mathsf{inr}(\star) = \mathsf{inl}(x)$, which is a contradiction.

□

This proof has been formalized [1] in Agda using the HoTT library.


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# References


[1] A. B. Booij. *Parametricity and excluded middle in Agda*. University of Birmingham, UK. URL: `http://www.cs.bham.ac.uk/~abb538/agda/nonparametric.html`.

[2] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: `http://homotopytypetheory.org/book`, 2013.