A Hofmann-Mislove theorem for bitopological spaces

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Abstract

We present a Stone duality for bitopological spaces in analogy to the duality between topological spaces and frames, and discuss the resulting notions of sobriety and spatiality. Under the additional assumption of regularity, we prove a characterisation theorem for subsets of a bisober space that are compact in one and closed in the other topology. This is in analogy to the celebrated Hofmann-Mislove theorem for sober spaces. We link the characterisation to Taylor's and Escardó's reading of the Hofmann-Mislove theorem as continuous quantification over a subspace.

Keywords: Bitopological spaces, d-frames, Stone duality, sober spaces, Hofmann-Mislove theorem

Introduction

The Hofmann-Mislove theorem states that in a sober space the open neighbourhood filters of compact saturated sets are precisely the Scott-open filters in the corresponding frame of opens. Mathematically, it has some remarkable consequences, such as the fact that the set of compact saturated subsets of

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a sober space form a dcpo when ordered by reverse inclusion, and it links Lawson duality (applied to the frame of opens) to the idea of the co-compact topology on the space, [19]. A modern and comprehensive presentation by the original authors can be found in section II-1 of [11].

The significance of the Hofmann-Mislove theorem in computer science took some time to emerge, and credit in this respect is due to Plotkin, [20, 21], Smyth, [22], and Vickers, [24], who pointed out that it is at the core of the proof that the upper powerdomain (defined as a free algebraic theory) has a concrete representation as a set of subsets of the given domain. Quite unexpectedly, it was also required in the classification of cartesian closed categories of domains, [15]. More recently, Taylor, [23], and Escardó, [8], have interpreted the theorem as expressing the idea that the compact saturated sets are precisely those for which there is a continuous universal quantifier. To this end, they read "open set" as "predicate" and "Scott-open filter of opens" as a map from predicates to Sierpiński space that is Scott-continuous and finite meet preserving, that is, as a "quantifier" which tells us whether a predicate is true for all elements of the corresponding compact set. Amazingly, such a quantifier exists not only in the mathematical model but can in fact be implemented in a *sequential* programming language, see [5, 10, 9].

Below we present a Stone duality for bitopological spaces motivated by the idea that a predicate may not only be true for some states, but in general will be false for others, and that the mechanisms for establishing falsehood will in general be different from those that establish truth. As Smyth has stressed, the positive extents of *observable* predicates form a topology, and so all we do is to add a second topology for the negative extents. However, in semantics we are already quite familiar with dealing with two topologies: Early on in the study of continuous lattices it was discovered by Lawson that the "weak lower topology" is a natural partner for the Scott-topology, their join being the (compact Hausdorff) Lawson topology. On hyperspaces $Y \subseteq \mathcal{P}X$ one naturally has the upper topology generated by sets of the form $\Box O := \{A \in Y \mid A \subseteq O\}$ (O an open in the original space), and the lower topology generated by sets of the form $\Diamond O := \{A \in Y \mid A \cap O \neq \emptyset\}$. Abramsky, [1], showed that the three powerdomains can be obtained systematically from this (bi-)topological point of view.

Our interest in bitopological spaces was driven by these examples and also by a desire to analyse various Stone dualities, but there is no room here to expand on this latter aspect; instead we refer the reader to the report [16].

The goal of the present paper is to exhibit a Hofmann-Mislove type theorem that, like its classical counterpart, admits a computational reading as a statement about quantifiers. The predicates to be quantified refer to Belnap's four-valued logic, [4], that is, in any state they can be true, false, unknown, or contradictory. To explain the effect of quantification let these four truth values be represented by $\{\text{true}\}$, $\{\text{false}\}$, $\{\}$, and $\{\text{true}, \text{false}\}$. Given a set A of states (which is a subset of a state space X) and a four-valued predicate φ, the result of quantification (i.e., $\forall x \in A.\, \varphi(x)$) will contain true if $\varphi(x)$ contains true for *all* $x \in A$; it will contain false if $\varphi(x)$ contains false for *some* $x \in A$. We note that $\forall x \in A.\, \varphi(x)$ could be false (and not contradictory) even if φ is contradictory for some $x \in A$. However, if φ is not contradictory for any state $x \in X$ then $\forall x \in A.\, \varphi(x)$ will also not be contradictory. Likewise, if φ is not unknown for any state x then $\forall x \in A.\, \varphi(x)$ will also not be unknown.

Before we can state and prove our Hofmann-Mislove Theorem, we must develop the necessary bitopological background of four-valued logic. We believe our approach to be novel, so the presentation is quite detailed. For comparison and reference we present the classical Hofmann-Mislove Theorem and its Stone duality context in Section 1, then introduce *d-frames* as a bitopological analogue of frames in Section 2. In Section 3 we demonstrate that in this setting Belnap's distinction between logical order and information order emerges naturally, and that there is an algebraic connection between the two. By analysing the spatial case, we postulate some "reasonable" requirements for d-frames in Section 4. Up to this point, most proofs are straightforward and mostly omitted. The theory of d-frames comes into its own once regularity is assumed, and Section 5, where we prove our Hofmann-Mislove Theorem, constitutes the mathematical core of the paper. In Section 6 we consider the dual concept of a continuous existential quantifier, and link the presence of continuous quantifiers to bitopological compactness.

There is only room to review classical Stone duality, so we have to assume that the reader is familiar with basic notions from topology, ordered sets, category theory, and Stone duality. For background reading on the first two topics we recommend [7], for the latter, [14]. Alternatively, either of the texts [11] and [2] also covers the necessary prerequisites.

A preliminary version of this paper appeared as [17].

1 Stone duality and the Hofmann-Mislove theorem

We briefly review the duality between topological spaces and frames. For more details see [2, Chapter 7], and [14, 11].

Definition 1.1 *A* frame *is a complete lattice in which finite meets distribute over arbitrary joins. We denote with* $\sqsubseteq$, $\sqcap$, $\bigsqcup$, 0*, and* 1 *the order, finite meets, arbitrary joins, least and largest element, respectively.*[1]

A frame homomorphism *preserves finite meets and arbitrary joins; thus we have the category* **Frm**.

For $(X;\tau)$ a topological space, $(\tau;\subseteq)$ is a frame; for $f\colon (X;\tau) \to (X';\tau')$ a continuous function, $f^{-1}\colon \tau' \to \tau$ is a frame homomorphism. These are the constituents of the contravariant functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$. It is represented by $\mathbf{Top}(-,\mathbb{S})$ where $\mathbb{S}$ is *Sierpiński space.*[2]

The collection $\mathcal{N}(a)$ of open neighbourhoods of a point a in a topological space $(X;\tau)$ forms a *completely prime filter* in the frame $\Omega\, X$, that is, it is an upper set, closed under finite intersections, and whenever $\bigcup \mathcal{O} \in \mathcal{N}(a)$ then $\mathcal{O} \cap \mathcal{N}(a) \neq \emptyset$. This leads one to consider the set of *points* (sometimes called "abstract points" for emphasis) of a frame L to be the collection $\mathsf{spec}\, L$ of completely prime filters. Abstract points are exactly the pre-images of $\{1\}$ under homomorphisms from L to $2 = \{0 < 1\}$.

A frame L induces a topology on $\mathsf{spec}\, L$ whose opens are of the form $\Phi(x) = \{F \in \mathsf{spec}\, L \mid x \in F\}$ with $x \in L$. A frame homomorphism $h\colon L \to L'$ induces a continuous function $\mathsf{spec}\, h\colon \mathsf{spec}\, L' \to \mathsf{spec}\, L$ by letting $\mathsf{spec}\, h(F) := h^{-1}(F)$ for $F \in \mathsf{spec}\, L'$. These are the components of the contravariant functor spec from **Frm** to **Top**, represented by $\mathbf{Frm}(-,2)$.

Theorem 1.2 *The functors* Ω *and* spec *constitute a dual adjunction between* **Top** *and* **Frm**.

The unit and co-unit of this adjunction are simply $\mathcal{N}$ and Φ. That is, for any space $(X;\tau)$ the map $\eta_X\colon X \to \mathsf{spec}\,\Omega\, X$, given by $a \mapsto \mathcal{N}(a)$, is continuous; it is also open onto its image. Likewise, for any frame L the map

[1] We use "square" symbols for the operations of a frame to distinguish them from the "logical" operations of a d-frame, to be introduced in Section 3.

[2] Sierpiński space has two points and precisely one non-trivial open set.

$\epsilon_L \colon L \to \Omega\, \mathsf{spec}\, L$, given by $x \mapsto \Phi(x)$, is a frame homomorphism; it is also surjective.

We can ask when a frame L is *spatial* in the sense that it is isomorphic to $\Omega\, X$ for some space X. As it turns out, there is a canonical candidate for X, namely, $\mathsf{spec}\, L$; more precisely, L is spatial if and only if ϵ_L is a frame isomorphism. Because ϵ_L is already a surjective frame homomorphism, this holds if and only if ϵ_L is injective.

Similarly, we can ask when a space X is *sober* in the sense that it is homeomorphic to $\mathsf{spec}\, L$ for some frame L. By the same reasoning as in frames, this holds if and only if η_X is a homeomorphism. Because η_X is already continuous and open onto its image, it suffices for η_X to be a bijection. Injectivity is precisely the T_0 axiom and surjectivity says that every completely prime filter of opens is the neighbourhood filter of a point.

Theorem 1.3 *The functors Ω and spec restrict to a dual equivalence between sober spaces and spatial frames.*

This is the setting for the *Hofmann-Mislove theorem*, [13], which we are now ready to state.

Theorem 1.4 *In a sober space (X, τ), there is a bijection between the set of compact saturated subsets of X and the set of Scott-open filters in τ.*

Although a direct proof is possible, [18], it is more useful for us to refer to Stone duality, as in the original paper [13]:

Lemma 1.5 *A Scott-open filter in a frame L is equal to the intersection of the collection of completely prime filters containing it.*

Proof. (Sketch) Let $\mathcal{S}$ be the Scott-open filter and a an element not in $\mathcal{S}$. Extend a to a maximal chain outside $\mathcal{S}$ and take its supremum v, which by Scott openness is a maximal element of $L \setminus \mathcal{S}$. Because $\mathcal{S}$ is a filter, v is meet irreducible, and because L is distributive, it is furthermore meet prime. It follows that the set $L \setminus {\downarrow} v$ is a completely prime filter that separates a from $\mathcal{S}$. ∎

Proof. (of 1.4) Clearly, the open neighbourhoods of a compact subset form a Scott-open filter in the lattice of open sets. For the converse, let A be the intersection of a Scott-open filter $\mathcal{S}$ of opens. By the lemma, every open neighbourhood of A belongs to $\mathcal{S}$. Because $\mathcal{S}$ is assumed to be Scott-open, A is compact (and obviously saturated).

A saturated set is the intersection of its open neighbourhoods by definition, and a Scott-open filter is the intersection of the completely prime filters containing it by the lemma, so the two assignments are inverses of each other. ∎

2 Stone duality for bitopological spaces

Without spending too much time on motivation, we now sketch a Stone duality for bitopological spaces; for the full picture we refer to [16].

A *bitopological space* is a set X together with two topologies τ_+ and τ_-. No connection between the two topologies is assumed. Morphisms between bitopological spaces are required to be continuous with respect to each of the two topologies; this gives rise to the category **biTop**.

For a Stone dual it is natural to consider pairs (L_+, L_-) of frames (and pairs of frame homomorphisms) but for some purposes it is more convenient to axiomatise the product $\tau_+ \times \tau_-$, that is, to have a single-sorted algebraic structure. In fact, the two views are entirely equivalent:

Proposition 2.1 *The category* **Frm**$\times$**Frm** *is equivalent to the category whose objects are frames which contain a pair of complemented elements* tt *and* ff*, and whose morphisms are frame homomorphisms that preserve* tt *and* ff*.*

Proof. In one direction, one assigns to a pair (L_+, L_-) the product $L_+ \times L_-$ and the constants $tt := (1, 0)$ and $ff := (0, 1)$. In the other direction, one assigns to $(L; tt, ff)$ the two frames $L_+ := [0, tt]$ and $L_- := [0, ff]$. The isomorphism from L to $[0, tt] \times [0, ff]$ is given by $\alpha \mapsto \langle \alpha_+, \alpha_- \rangle := \langle \alpha \sqcap tt, \alpha \sqcap ff \rangle$. The isomorphism from $L_+ \times L_-$ to L is given by $\langle x, y \rangle \mapsto x \sqcup y$. ∎

In addition to the notation $\langle \alpha_+, \alpha_- \rangle$ introduced in the proof above we will also use $\alpha \sqsubseteq_+ \beta$ in case $\alpha_+ \sqsubseteq \beta_+$, and similarly $\sqsubseteq_-$. One has $\alpha \sqsubseteq \beta$ if and only if $\alpha \sqsubseteq_+ \beta$ and $\alpha \sqsubseteq_- \beta$.

Having two frames is not enough, however, as we also need to express the fact that they represent topologies *on the same set*. One approach for achieving this was introduced by Banaschewski, Brümmer, and Hardie in [3]; their *biframes* axiomatise the two topologies and the joint refinement $\tau_+ \vee \tau_-$.

Our proposal is different; we only record when two open sets $O_+ \in \tau_+$ and $O_- \in \tau_-$ are disjoint from each other, and when they cover the whole space X. In the first case we say that they are *consistent*, in the second that they are *total*.

Definition 2.2 *A* d-frame *consists of a frame L, a pair of complemented elements tt and ff, and two unary predicates* con *and* tot. *Morphisms between d-frames are required to preserve all of this structure. The resulting category is denoted by* **dFrm**.

As we have already explained informally, the contravariant functor Ω from bitopological spaces to d-frames assigns to a space $(X;\tau_+,\tau_-)$ the d-frame $(\tau_+\times\tau_-;(X,\emptyset),(\emptyset,X),\mathsf{con},\mathsf{tot})$ where $(U,V)\in\mathsf{con}$ if and only if $U\cap V=\emptyset$ and $(U,V)\in\mathsf{tot}$ if and only if $U\cup V=X$. The functor associates with a bicontinuous function f the map $(U,V)\mapsto(f^{-1}(U),f^{-1}(V))$. A trivial bit of set theory will convince the reader that the consistency and totality predicates are preserved. Figure 1 shows some small examples. The bitopological space $\mathbb{S}.\mathbb{S}$, which looks like a product of two copies of Sierpiński space, allows us to represent the functor Ω as $\mathbf{biTop}(-,\mathbb{S}.\mathbb{S})$. Note how the four elements of $\mathbb{S}.\mathbb{S}$ correspond to the four ways in which an element of the space can be related to an open from τ_+ and an open from τ_-: it can be in one of the two but not the other, it can be in both, or it can be in neither.

For a functor in the reverse direction, we continue to follow the theory of frames by employing 2.2, depicted in the upper right corner of Figure 1, as the dualising object. More precisely:

Definition 2.3 *The d-frame* 2.2 *is based on the frame* 2×2 *where* 2 *is the two-element frame with elements* $0<1$. *Its smallest element is* $(0,0)$, *denoted again by* 0, *and the largest element is* $(1,1)$ *and denoted by* 1. *The other two elements are* $tt=(1,0)$ *and* $ff=(0,1)$. *The consistency predicate contains* 0, tt, *and* ff, *the totality predicate* 1, tt, *and* ff.

Note that 2.2 is isomorphic to $\Omega(*)$ where $*$ is the bitopological space with one point.

Definition 2.4 *A* d-point *of a d-frame $\mathcal{L}$ is a* **dFrm** *morphism from $\mathcal{L}$ to* 2.2. *When it is clear that we are within the context of d-frames we usually drop the prefix "d-" from "d-point."*

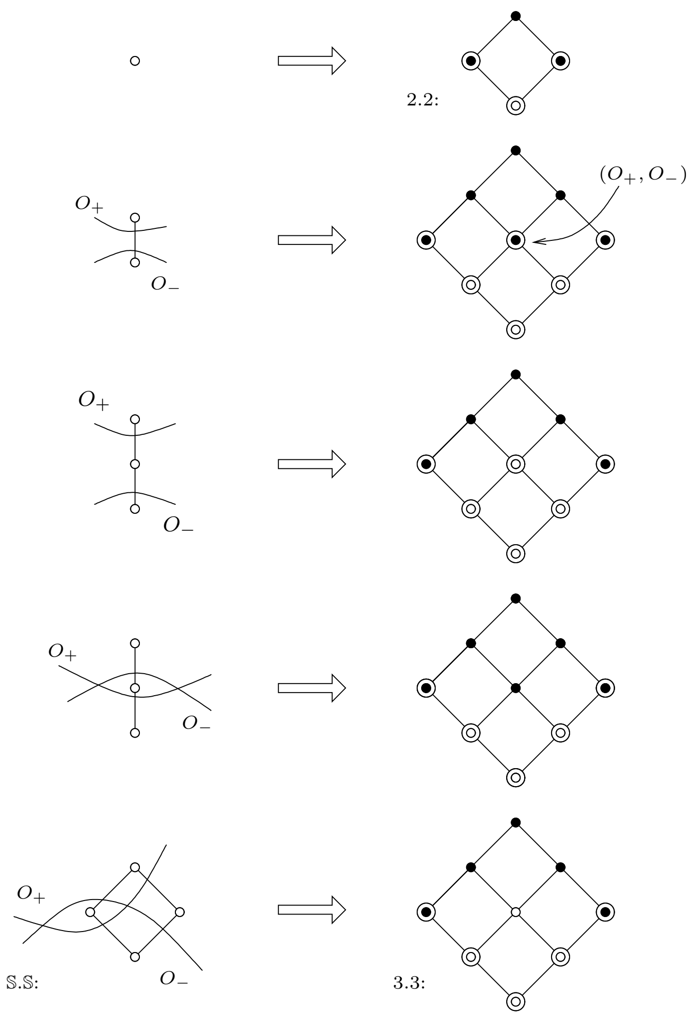

Figure 1: Some bitopological spaces and their concrete d-frames. (D-frame elements in the con-predicate are indicated by an additional circle, those in the tot-predicate are filled in.)

A point $p\colon \mathcal{L} \to 2.2$ is completely determined by the pre-images $F_+^* := p^{-1}(\{\mathit{tt}, 1\})$ and $F_-^* := p^{-1}(\{\mathit{ff}, 1\})$. These two subsets of $\mathcal{L}$ have the following properties:

- F_+^* and F_-^* are completely prime filters;
- $\mathit{tt} \in F_+^*$ and $\mathit{ff} \in F_-^*$;
- $\alpha \in \mathsf{con} \quad\Rightarrow\quad \alpha \notin F_+^*$ or $\alpha \notin F_-^*$;
- $\alpha \in \mathsf{tot} \quad\Rightarrow\quad \alpha \in F_+^*$ or $\alpha \in F_-^*$.

It is easy to see that a pair of subsets with these characteristics gives rise to a point, in other words, this is an alternative description of the concept of a point for a d-frame.

Yet another formulation is obtained by employing Proposition 2.1 and viewing $\mathcal{L}$ as a product of two frames L_+ and L_-, 2.2 as the product of two copies of 2, and p as a pair of homomorphisms $p_+\colon L_+ \to 2$, $p_-\colon L_- \to 2$. By taking pre-images, we obtain subsets $F_+ := p_+^{-1}(1)$ and$F_- := p_-^{-1}(1)$. which satisfy:

- F_+ and F_- are completely prime filters of L_+ and L_-, respectively;
- ($\mathrm{dp}_{\mathsf{con}}$) $\alpha \in \mathsf{con} \quad\Rightarrow\quad \alpha_+ \notin F_+$ or $\alpha_- \notin F_-$;
- ($\mathrm{dp}_{\mathsf{tot}}$) $\alpha \in \mathsf{tot} \quad\Rightarrow\quad \alpha_+ \in F_+$ or $\alpha_- \in F_-$.

The connection between con, tot, F_+^*, F_-^*, F_+, and F_- is illustrated in Figure 2.

For the purposes of the present paper we found the last formulation to be the most appropriate one.

The set of points of a d-frame becomes a bitopological space by considering the collection of $\Phi_+(x) := \{(F_+, F_-) \mid x \in F_+\}$, $x \in L_+$, as the first topology $\mathcal{T}_+$, and the collection of $\Phi_-(y) := \{(F_+, F_-) \mid y \in F_-\}$, $y \in L_-$, as the second topology $\mathcal{T}_-$. Together, this is the *spectrum* of the d-frame $\mathcal{L}$, which we denote as $\mathsf{spec}\,\mathcal{L}$, mirroring the notation for frames. The construction for objects is extended to a (contravariant) functor $\mathsf{spec}\colon \mathbf{dFrm} \to \mathbf{biTop}$ in the usual way. The proof of the following is now completely analogous to the single frame case.

Theorem 2.5 *The functors* Ω *and* spec *establish a dual adjunction between* **biTop** *and* **dFrm**.

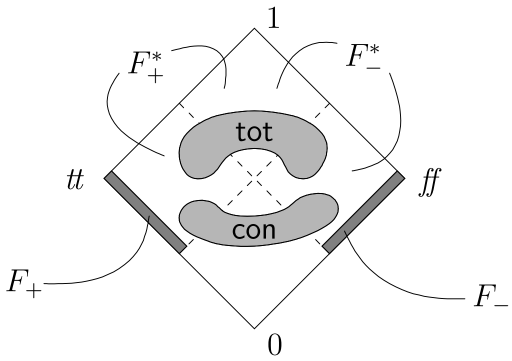

Figure 2: An abstract point in a d-frame.

We say that a bitopological space X is *(d-) sober* if it is bihomeomorphic to $\mathsf{spec}\,\mathcal{L}$ for some d-frame $\mathcal{L}$; this is equivalent to the unit $x \mapsto (\mathcal{N}_+(x), \mathcal{N}_-(x))$ being a bijection.

Example 2.6 *All the bitopological spaces in Figure 1 are d-sober. For the one-point space this is clear, as the associated d-frame admits only one point. For the other four spaces one argues as follows: The underlying frame is the same in each case and it admits four completely prime filters:*

$$\begin{array}{lclclcl} F_+^1 & := & \uparrow tt & & F_-^1 & := & \uparrow ff \\ F_+^2 & := & \uparrow(O_+, \emptyset) & & F_-^2 & := & \uparrow(\emptyset, O_-) \end{array}$$

The notation already indicates which of these can be used as the first, respectively second, component of a point (as prescribed by the requirements $tt \in F_+^*$, $ff \in F_-^*$). From this we get four possible combinations, and these are indeed all available in the last example. In the other three examples, the $\mathsf{con}/\mathsf{tot}$ *labelling of the element (O_+, O_-) in the centre of the d-frame excludes certain combinations: if it belongs to* con*, then F_+^2 cannot be paired with F_-^2, and if it belongs to* tot *then F_+^1 cannot be paired with F_-^1.*

For an exploration into the concept of d-sobriety we refer to [16]; here we confine ourselves to one particular class of examples.

Definition 2.7 *A bitopological space $(X; \tau_+, \tau_-)$ is called* order-separated *if $\leq\, = \,\leq_+ \cap \geq_-$ is a partial order and $x \not\leq y$ implies that there are disjoint open sets $O_+ \in \tau_+$ and $O_- \in \tau_-$ such that $x \in O_+$ and $y \in O_-$. (The relations $\leq_+$ and $\leq_-$ refer to the specialisation orders on X with respect to τ_+ and τ_-, respectively.)*

Lemma 2.8 *In an order-separated bitopological space the following are true:*

1. $\leq_+ \;=\; \geq_- \;=\; \leq$;
2. $\leq_+ \cap \leq_- \;=\;$ '$=$'.

Proof. (1) For the first equality assume $x \not\leq_+ y$. This implies $x \not\leq y$ and we get a separating consistent pair (O_+, O_-). Since $y \in O_-$ but $x \notin O_-$ we conclude $x \not\geq_- y$. So $\not\leq_+ \;=\; \not\geq_-$ and this is equivalent to $\leq_+ \;=\; \geq_-$. The second equality follows by the definition of $\leq$.

(2) From the first part we get $\leq_+ \cap \leq_- \;=\; \leq \cap \geq$ and the claim then follows from anti-symmetry of $\leq$. ∎

Theorem 2.9 *Order-separated bitopological spaces are sober.*

Proof. Order separation clearly implies that the canonical map $\eta\colon X \to \mathsf{spec}\,\Omega\, X$ is injective; the real issue is surjectivity. So assume that (F_+, F_-) is a point of $\Omega\, X$. Consider the two sets

$$V_+ := \bigcup \{O_+ \in \tau_+ \mid O_+ \notin F_+\} \qquad V_- := \bigcup \{O_- \in \tau_- \mid O_- \notin F_-\}$$

and their complements V_+^c, V_-^c. Because of condition ($\mathrm{dp}_{\mathsf{tot}}$), $V_+ \cup V_-$ cannot be the whole space, in other words, the intersection $V_+^c \cap V_-^c$ is non-empty.

Next we show that every element of V_+^c is below every element of V_-^c in the specialisation order $\leq \;=\; \leq_+ \;=\; \geq_-$. Indeed, if $x \in V_+^c$, $y \in V_-^c$, and $x \not\leq y$, then by order separation there is a pair (O_+, O_-) with $O_+ \cap O_- = \emptyset$, $x \in O_+$, and $y \in O_-$. By definition of V_+, V_- we have $O_+ \in F_+$ and $O_- \in F_-$, contradicting condition ($\mathrm{dp}_{\mathsf{con}}$) of d-points.

Finally, let a be an element in the intersection $V_+^c \cap V_-^c$. We show that F_+ is the neighbourhood filter of a in τ_+. Assume $a \in O_+$; this implies $O_+ \not\subseteq V_+$ and the latter is equivalent to $O_+ \in F_+$. For the converse we start at $O_+ \not\subseteq V_+$, which gives us an element $b \in V_+^c \cap O_+$ about which we already know that $b \leq a$. It follows that $b \leq_+ a$ and hence $a \in O_+$. ∎

From this result it follows immediately that the real line together with the usual upper and lower topology is d-sober. Likewise, one sees that the *punctured unit interval* $[0, 1] \setminus \{\frac{1}{2}\}$ is d-sober with respect to the same two topologies. Note that neither is sober in the traditional sense when equipped with only one of the topologies.

3 The logical structure of d-frames

Before we consider spatiality for d-frames let us have a look at the duality from the point of view of logic. For this we interpret the elements of a d-frame $\mathcal{L}$ as *logical propositions*. An abstract point (F_+^*, F_-^*) is then a *model*, and F_+^* consists of those propositions which are true in the model, F_-^* of those that are false. If a proposition belongs to con then for no model is it both true and false (and may be neither); if it belongs to tot then in every model it is either true or false (or indeed both). The set of all models (i.e., $\mathsf{spec}\,\mathcal{L}$) becomes a bitopological space by collecting into one topology all sets of models in which some proposition is true (the "positive extents") and in the other the sets of models where some proposition is false (the "negative extents").

From this perspective it is natural to consider an order between propositions which increases the positive extent and shrinks the negative one. As it turns out, this additional relation is always present in a d-frame, and in fact it follows from the distributive lattice structure and the two complemented elements alone. The earliest reference to this appears to be [6], but the proof is entirely straightforward and can be left as an exercise.

Proposition 3.1 *Let $(L; \sqcap, \sqcup, 1, 0)$ be a bounded distributive lattice, and (t, f) a complemented pair in L, that is, $t \sqcap f = 0$ and $t \sqcup f = 1$. Then by defining*

$$\begin{aligned} x \wedge y &:= (x \sqcap f) \sqcup (y \sqcap f) \sqcup (x \sqcap y) = (x \sqcup f) \sqcap (y \sqcup f) \sqcap (x \sqcup y) \\ x \vee y &:= (x \sqcup t) \sqcap (y \sqcup t) \sqcap (x \sqcup y) = (x \sqcap t) \sqcup (y \sqcap t) \sqcup (x \sqcap y) \end{aligned}$$

one obtains another bounded distributive lattice $(L; \wedge, \vee, t, f)$, in which $(1, 0)$ is a complemented pair. The original operations are recovered from it as

$$\begin{aligned} x \sqcap y &= (x \wedge 0) \vee (y \wedge 0) \vee (x \wedge y) = (x \vee 0) \wedge (y \vee 0) \wedge (x \vee y) \\ x \sqcup y &= (x \vee 1) \wedge (y \vee 1) \wedge (x \vee y) = (x \wedge 1) \vee (y \wedge 1) \vee (x \wedge y) \end{aligned}$$

Furthermore, any two of the operations $\sqcap$, $\sqcup$, $\wedge$, and $\vee$ distribute over each other. If L is a frame, then $\wedge$ and $\vee$ are also Scott continuous.

This justifies our choice of symbols tt and ff in a d-frame, and suggests that we regard $(L; \wedge, \vee, tt, ff)$ as the *logical structure* of a d-frame. Altogether, then, we see that d-frames are special "bilattices," which were introduced by Ginsberg, [12], as a generalisation of Belnap's four-valued logic [4].

Indeed, the four-element d-frame 2.2, which we introduced in Definition 2.4 and used as the dualising object in **dFrm**, is exactly Belnap's lattice of truth values, except that we added the consistency and totality predicates.

Exploiting Proposition 2.1 we can easily compute conjunction and disjunction in terms of the representation of a d-frame as $L_+ \times L_-$:

$$\begin{array}{rcl} \langle x, y\rangle \wedge \langle x', y'\rangle & := & \langle x \sqcap x', y \sqcup y'\rangle \\ \langle x, y\rangle \vee \langle x', y'\rangle & := & \langle x \sqcup x', y \sqcap y'\rangle \end{array}$$

Note the reversal of order in the second component. This makes sense, as we think of the second frame as providing negative answers.

4 Reasonable d-frames and spatiality

We say that a d-frame $\mathcal{L}$ is *spatial* if it is isomorphic to $\Omega\, X$ for some bitopological space X. This is equivalent to the co-unit $\epsilon\colon \alpha \mapsto (\Phi_+(\alpha), \Phi_-(\alpha))$ being an isomorphism of d-frames. As it is always surjective by definition, the condition boils down to ϵ being injective and reflecting con and tot. If this is spelt out concretely, one arrives at the following:

Proposition 4.1 *A d-frame $\mathcal{L}$ is spatial if and only if the following four conditions are satisfied:*

$$\begin{array}{ll} (\mathrm{s}_+) & \forall \alpha \not\sqsubseteq_+ \beta\ \ \exists (F_+, F_-) \in \mathsf{spec}\,\mathcal{L}.\ \ \alpha_+ \in F_+, \beta_+ \notin F_+; \\ (\mathrm{s}_-) & \forall \alpha \not\sqsubseteq_- \beta\ \ \exists (F_+, F_-) \in \mathsf{spec}\,\mathcal{L}.\ \ \alpha_- \in F_-, \beta_- \notin F_-; \\ (\mathrm{s}_{\mathsf{con}}) & \forall \alpha \notin \mathsf{con}\ \ \exists (F_+, F_-) \in \mathsf{spec}\,\mathcal{L}.\ \ \alpha_+ \in F_+, \alpha_- \in F_-; \\ (\mathrm{s}_{\mathsf{tot}}) & \forall \alpha \notin \mathsf{tot}\ \ \exists (F_+, F_-) \in \mathsf{spec}\,\mathcal{L}.\ \ \alpha_+ \notin F_+, \alpha_- \notin F_-; \end{array}$$

The following lemma is very easy to prove for concrete d-frames that arise from a bitopological space, and it confirms the intuition of con as the set of pairs of open sets that do not intersect, and tot as those pairs that cover the whole space.

Lemma 4.2 *Let $(L; tt, f\!f; \mathsf{con}, \mathsf{tot})$ be a spatial d-frame. The following properties hold:*

$$\begin{array}{lrcl} (\mathsf{con}\text{-}\!\downarrow) & \alpha \sqsubseteq \beta\ \&\ \beta \in \mathsf{con} & \Longrightarrow & \alpha \in \mathsf{con} \\ (\mathsf{tot}\text{-}\!\uparrow) & \alpha \sqsubseteq \beta\ \&\ \alpha \in \mathsf{tot} & \Longrightarrow & \beta \in \mathsf{tot} \end{array}$$

(con$-tt$)	$tt \in$ con
(con$-ff$)	$ff \in$ con
(con$-\wedge$)	$\alpha \in$ con & $\beta \in$ con $\Longrightarrow$ $(\alpha \wedge \beta) \in$ con
(con$-\vee$)	$\alpha \in$ con & $\beta \in$ con $\Longrightarrow$ $(\alpha \vee \beta) \in$ con
(tot$-tt$)	$tt \in$ tot
(tot$-ff$)	$ff \in$ tot
(tot$-\wedge$)	$\alpha \in$ tot & $\beta \in$ tot $\Longrightarrow$ $(\alpha \wedge \beta) \in$ tot
(tot$-\vee$)	$\alpha \in$ tot & $\beta \in$ tot $\Longrightarrow$ $(\alpha \vee \beta) \in$ tot
(con$-\bigsqcup^{\uparrow}$)	$A \subseteq$ con *directed w.r.t.* $\sqsubseteq$ $\Longrightarrow$ $\bigsqcup^{\uparrow} A \in$ con
(con$-$tot)	$\alpha \in$ con, $\beta \in$ tot, $(\alpha =_{+} \beta$ *or* $\alpha =_{-} \beta)$ $\Longrightarrow$ $\alpha \sqsubseteq \beta$

Definition 4.3 *A d-frame which satisfies the properties stated in Lemma 4.2 is called* reasonable*. The category of reasonable d-frames is denoted by* **rdFrm**.

Note that the converse of Lemma 4.2 does not hold, i.e., a reasonable d-frame need not be spatial: take a frame L without any points and consider $(L \times L; (1,0), (0,1), \mathsf{con}, \mathsf{tot})$ where $\langle x, y \rangle \in \mathsf{con}$ if $x \sqcap y = 0$, and $\langle x, y \rangle \in \mathsf{tot}$ if $x \sqcup y = 1$. It is a trivial exercise to prove that the resulting d-frame is reasonable, but it obviously can't have any points.

Proposition 4.4 *The forgetful functor from* **rdFrm** *to* **Set** *has a left adjoint.*

Proof. The free reasonable d-frame over a set A is $(FA \times FA; (1,0), (0,1), \mathsf{con}, \mathsf{tot})$ where FA is the free frame over A. Generators are the pairs (a, a), $a \in A$. The two relations are chosen minimally: $\langle x, y \rangle \in \mathsf{con}$ if and only if $x = 0$ or $y = 0$; $\langle x, y \rangle \in \mathsf{tot}$ if and only if $x = 1$ or $y = 1$. The conditions for a reasonable d-frame are proved by case analysis. ∎

As an example, the structure labelled 3.3 in Figure 1 is the free reasonable d-frame generated by a one-element set.

The following additional property of spatial d-frames will also play a part in our presentation of a Hofmann-Mislove theorem for sober bitopological spaces, but we do not consider it elementary enough to be included in the definition of "reasonable." The proof-theoretic terminology used in its label refers to a presentation of d-frames that places more emphasis on the logical structure, see [16, Section 7].

Proposition 4.5 *Every spatial d-frame satisfies the following property:*

$$\text{(CUT)} \qquad \left.\begin{array}{c} \forall i \in I.\ \langle a_i, b_i\rangle \in \mathsf{con} \\ \forall i \in I.\ \langle x \sqcup a_i, y\rangle \in \mathsf{tot} \\ \langle x, y \sqcup \bigsqcup_{i\in I} b_i\rangle \in \mathsf{tot} \end{array}\right\} \implies \langle x, y\rangle \in \mathsf{tot}$$

5 Regularity and the Hofmann-Mislove theorem

A major practical problem with d-frames is that it is very difficult to construct points for them. For example, consider the proof of the Hofmann-Mislove Lemma 1.5, where we exploited the fact that in a frame there is a one-to-one correspondence between completely prime filters F and $\sqcap$-prime elements v (given by the assignments $v \mapsto L \setminus \downarrow v$ and $F \mapsto \bigsqcup L \setminus F$). The analogue for d-frames is not very helpful. The situation improves considerably if we also require regularity.

Definition 5.1 *Let* $(L; tt, ff; \mathsf{con}, \mathsf{tot})$ *be a reasonable d-frame. For two elements* $x, x' \in L_+$ *we say that* x' *is* well-inside x *(and write* $x' \lhd x$*) if there is* $y \in L_-$ *such that* $\langle x', y\rangle \in \mathsf{con}$ *and* $\langle x, y\rangle \in \mathsf{tot}$*. To avoid lengthy verbiage, we will usually write* $r_{x' \lhd x}$ *for the "witnessing" element* y *(although it is not uniquely determined). On* L_- *the well-inside relation is defined analogously.*

A d-frame is called regular *if every* $x \in L_+$ *is the supremum of elements well-inside it, and similarly for elements of* L_-*.*

For a bitopological space to be regular we require that at least one of the two topologies is T_0 *and that the corresponding d-frame is regular.*[3]

We note that the elements well-inside a fixed element x of a reasonable d-frame form a directed set; this follows from ($\mathsf{con}-\vee$) and ($\mathsf{tot}-\vee$). That they are all below x is a consequence of ($\mathsf{con}-\mathsf{tot}$). $1 \lhd 1$ is always true as 0 can be chosen as the witness in the other frame. It is an easy exercise to show that a regular bitopological space is order-separated (and hence d-sober), but a regular d-frame need not be spatial.

Lemma 5.2 *Let* $\mathcal{L}$ *be a reasonable d-frame and* $x \in L_+$*. Define*

$$\mathsf{P}(x) := \{b \in L_- \mid \exists a \not\sqsubseteq x.\ \langle a, b\rangle \in \mathsf{con}\} \quad \textit{and} \quad \mathsf{C}(x) := \{b \in L_- \mid \langle x, b\rangle \notin \mathsf{tot}\}$$

[3]It then follows that the other topology is T_0 as well.

1. $\mathsf{P}(x) \subseteq \mathsf{C}(x)$;

2. *If* $\mathcal{L}$ *is regular then* $\bigsqcup \mathsf{P}(x) = \bigsqcup \mathsf{C}(x)$.

Proof. (1) is a direct consequence of (con−tot): if we have $\langle a, b\rangle \in \mathsf{con}$ and $\langle x, b\rangle \in \mathsf{tot}$ then $a \sqsubseteq x$ follows.

For (2) let $b' \lhd b \in \mathsf{C}(x)$. The witness $r_{b'\lhd b}$ cannot be below x as otherwise we could conclude $\langle x, b\rangle \in \mathsf{tot}$ from $\langle r_{b'\lhd b}, b\rangle \in \mathsf{tot}$. We also have $\langle r_{b'\lhd b}, b'\rangle \in \mathsf{con}$ and so find that $b' \in \mathsf{P}(x)$. By regularity, $\bigsqcup \mathsf{P}(x)$ is above b itself. It follows that $\bigsqcup \mathsf{P}(x) \sqsupseteq \bigsqcup \mathsf{C}(x)$, and by (1) the two suprema are in fact the same. ∎

Lemma 5.3 *Let* $\mathcal{L}$ *be a reasonable d-frame and* $v_+ \in L_+$, $v_- \in L_-$. *Consider the following statements:*

(i) $v_- = \max \mathsf{C}(v_+)$ *and* $v_+ = \max \mathsf{C}(v_-)$;

(ii) $(L_+ \setminus \downarrow v_+, L_- \setminus \downarrow v_-)$ *is a d-point;*

(iii) $\langle v_+, v_-\rangle \notin \mathsf{tot}$ *and* $v_- \sqsupseteq \bigsqcup^{\uparrow} \mathsf{P}(v_+)$;

(iv) $\langle v_+, v_-\rangle$ *is a maximal element of* $(L_+ \times L_-) \setminus \mathsf{tot}$.

The following are true:

1. *(i)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii), and (i)* $\Rightarrow$ *(iv).*

2. *If* $\mathcal{L}$ *is regular then (iii)* $\Rightarrow$ *(i).*

3. *If* $\mathcal{L}$ *satisfies the* (CUT) *rule then (iv)* $\Rightarrow$ *(ii).*

Proof. Part (1), (i) $\Rightarrow$ (ii): If $\langle x, y\rangle \in \mathsf{tot}$ then either $x \not\sqsubseteq v_+$ or $y \not\sqsubseteq v_-$ as otherwise we would have $\langle v_+, v_-\rangle \in \mathsf{tot}$ by (tot−↑). If $\langle x, y\rangle \in \mathsf{con}$ and $x \not\sqsubseteq v_+$ then $y \in \mathsf{P}(v_+) \subseteq \mathsf{C}(v_+)$ by the previous lemma; hence $y \sqsubseteq v_-$. Thus we have shown that the pair $(L_+ \setminus \downarrow v_+, L_- \setminus \downarrow v_-)$ satisfies conditions ($\mathrm{dp}_{\mathsf{tot}}$) and ($\mathrm{dp}_{\mathsf{con}}$) for d-points and it remains to show that we have two completely prime filters. This will hold if v_+ and v_- are $\sqcap$-irreducible. So let $v_- = y \sqcap y'$; by (tot−∨) either $\langle v_+, y\rangle \notin \mathsf{tot}$ or $\langle v_+, y'\rangle \notin \mathsf{tot}$, which means that either $y = v_-$ or $y' = v_-$.

(ii) $\Rightarrow$ (iii): $\langle v_+, v_-\rangle \notin \mathsf{tot}$ follows from ($\mathrm{dp}_{\mathsf{tot}}$). For the second statement, if $x \not\sqsubseteq v_+$ and $\langle x, y\rangle \in \mathsf{con}$ then $y \sqsubseteq v_-$ by ($\mathrm{dp}_{\mathsf{con}}$). So we have $v_- \sqsupseteq \bigsqcup \mathsf{P}(v_+)$.

The set $\mathsf{P}(v_+)$ is directed because $L_+ \setminus \downarrow v_+$ is a filter and $(\mathsf{con}{-}\wedge)$ is assumed for reasonable d-frames.

(i) $\Rightarrow$ (iv) is trivial.

Part (2), (iii) $\Rightarrow$ (i): On the side of L_- we already have $v_- \sqsupseteq \bigsqcup \mathsf{C}(v_+)$ by the previous lemma. For L_+, assume $x \not\sqsubseteq v_+$. By regularity there is $x' \lhd x$ with $x' \not\sqsubseteq v_+$. Because of $\langle x', r_{x' \lhd x}\rangle \in \mathsf{con}$ we have $r_{x' \lhd x} \sqsubseteq v_-$ by assumption, and then from $\langle x, r_{x' \lhd x}\rangle \in \mathsf{tot}$ we infer $\langle x, v_-\rangle \in \mathsf{tot}$ by $(\mathsf{tot}{-}\uparrow)$. It follows that $\mathsf{C}(v_-) \subseteq \downarrow v_+$. Together with $\langle v_+, v_-\rangle \notin \mathsf{tot}$ this is exactly (i).

Part (3), (iv) $\Rightarrow$ (ii): As in (i) $\Rightarrow$ (ii) we get that v_+ and v_- are $\sqcap$-prime, and that condition $(\mathrm{dp}_{\mathsf{tot}})$ is satisfied for $(L_+ \setminus \downarrow v_+, L_- \setminus \downarrow v_-)$. In order to show $(\mathrm{dp}_{\mathsf{con}})$ assume $\langle x, y\rangle \in \mathsf{con}$. If we had $x \not\sqsubseteq v_+$ and $y \not\sqsubseteq v_-$ then by (the contrapositive of) the (CUT) rule we would have either $\langle v_+, v_- \sqcup y\rangle \notin \mathsf{tot}$ or $\langle v_+ \sqcup x, v_-\rangle \notin \mathsf{tot}$, contradicting the maximality of $\langle v_+, v_-\rangle$. ∎

We are ready to formulate and prove the d-frame analogue to the Hofmann-Mislove Lemma 1.5:

Lemma 5.4 *Let $\mathcal{L}$ be a regular d-frame. Assume that $\mathcal{S}_+$ is a Scott-open filter in L_+ and $\mathcal{U}_- = L_- \setminus \downarrow u_-$ is a completely prime upper set in L_- such that:*

$$(\mathrm{hm}_{\mathsf{con}}) \quad \alpha \in \mathsf{con} \implies \alpha_+ \notin \mathcal{S}_+ \text{ or } \alpha_- \notin \mathcal{U}_-$$
$$(\mathrm{hm}_{\mathsf{tot}}) \quad \alpha \in \mathsf{tot} \implies \alpha_+ \in \mathcal{S}_+ \text{ or } \alpha_- \in \mathcal{U}_-$$

Then the following are true:

1. $u_- = \bigsqcup^{\uparrow}\{b \mid \exists a \in \mathcal{S}_+. \langle a, b\rangle \in \mathsf{con}\}$, *that is, $\mathcal{U}_-$ is uniquely determined by $\mathcal{S}_+$.*
2. $\mathcal{S}_+ = \{a \mid \langle a, u_-\rangle \in \mathsf{tot}\}$, *that is, $\mathcal{S}_+$ is uniquely determined by $\mathcal{U}_-$.*
3. $x \sqsubseteq \mathcal{S}_+ \;\Leftrightarrow\; (x, u_-) \in \mathsf{con}$.
4. *For any point $(F_+, F_-) \in \mathsf{spec}\,\mathcal{L}$,* $\quad \mathcal{S}_+ \subseteq F_+ \;\Leftrightarrow\; F_- \subseteq \mathcal{U}_-$.

If $\mathcal{L}$ satisfies (CUT)*, then furthermore the following are true:*

5. *$\mathcal{S}_+$ is the intersection of all F_+ where (F_+, F_-) is a point and $\mathcal{S}_+ \subseteq F_+$.*
6. *$\mathcal{U}_-$ is the union of all F_- where (F_+, F_-) is a point and $F_- \subseteq \mathcal{U}_-$.*
7. *The set $A := \{(F_+, F_-) \mid \mathcal{S}_+ \subseteq F_+\} = \{(F_+, F_-) \mid F_- \subseteq \mathcal{U}_-\}$ is $\mathcal{T}_+$-compact saturated and $\mathcal{T}_-$-closed in the bitopological space $(\mathsf{spec}\,\mathcal{L}; \mathcal{T}_+, \mathcal{T}_-)$.*

Proof. (1) The element u_- can not be any smaller because of (hm$_{\mathsf{con}}$). For the converse assume $y \lhd u_-$. The corresponding witness $r_{y \lhd u_-}$ belongs to $\mathcal{S}_+$ by (hm$_{\mathsf{tot}}$) and so $y \in \{b \mid \exists a \in \mathcal{S}_+ .\, \langle a,b \rangle \in \mathsf{con}\}$. By regularity, then, $u_- \sqsubseteq \bigsqcup^{\uparrow}\{b \mid \exists a \in \mathcal{S}_+ .\, \langle a,b \rangle \in \mathsf{con}\}$.

(2) Because of (hm$_{\mathsf{tot}}$) it is clear that $\mathcal{S}_+$ must contain all $a \in L_+$ with $\langle a, u_- \rangle \in \mathsf{tot}$. For the converse let $x \in \mathcal{S}_+$. By regularity and Scott-openness of $\mathcal{S}_+$ there is $x' \lhd x$ still in $\mathcal{S}_+$. The corresponding witness $r_{x' \lhd x}$ must be below u_- because of (hm$_{\mathsf{con}}$), but then $\langle x, u_- \rangle \in \mathsf{tot}$ by ($\mathsf{tot}{-}\uparrow$).

(3) Assume $x \sqsubseteq a$ for all $a \in \mathcal{S}_+$. By ($\mathsf{con}{-}\downarrow$) we have $(x,b) \in \mathsf{con}$ for all $b \in \{b \mid \exists a \in \mathcal{S}_+ .\, (a,b) \in \mathsf{con}\}$, so $(x, u_-) \in \mathsf{con}$ by ($\mathsf{con}{-}\bigsqcup^{\uparrow}$) and part (1). For the converse, remember that $(a, u_-) \in \mathsf{tot}$ for all $a \in \mathcal{S}_+$ by (2), so $(x, u_-) \in \mathsf{con}$ implies $x \sqsubseteq a$ by ($\mathsf{con}{-}\mathsf{tot}$).

(4) We use $v_+ := \bigsqcup(L_+ \setminus F_+)$ and $v_- := \bigsqcup(L_- \setminus F_-)$ to synchronise notation with Lemma 5.3. Note that $\mathcal{S}_+ \subseteq F_+$ is equivalent to $v_+ \notin \mathcal{S}_+$, and $F_- \subseteq \mathcal{U}_-$ is equivalent to $u_- \sqsubseteq v_-$.

From $\mathcal{S}_+ \subseteq F_+$, 5.3(iii), and (1) we get $u_- \sqsubseteq v_-$ and hence $F_- \subseteq \mathcal{U}_-$. Starting with the right hand side, $F_- \subseteq \mathcal{U}_-$, we get $\mathsf{P}(v_-) \cap \mathcal{S}_+ = \emptyset$ by (hm$_{\mathsf{con}}$). So $v_+ = \bigsqcup^{\uparrow}\mathsf{P}(v_-) \notin \mathcal{S}_+$ and hence $\mathcal{S}_+ \subseteq F_+$.

(5) Assume that $x \notin \mathcal{S}_+$. Because $\mathcal{S}_+$ is assumed to be Scott-open, we can apply Zorn's Lemma to obtain a maximal element v_+ above x that does not belong to $\mathcal{S}_+$. The set $F_+ := L_+ \setminus {\downarrow} v_+$ is a completely prime filter that separates x from $\mathcal{S}_+$, and it remains to show that it is the first component of a d-point. According to Lemma 5.3 the right candidate is $F_- = L_- \setminus {\downarrow} v_-$ where $v_- = \bigsqcup^{\uparrow}\mathsf{P}(v_+) = \bigsqcup \mathsf{C}(v_+)$. Note that $u_- \sqsubseteq v_-$ as $u_- \in \mathsf{C}(v_+)$ by (hm$_{\mathsf{tot}}$). Using Lemma 5.3(iii) we only need to show that $\langle v_+, v_- \rangle \notin \mathsf{tot}$. For this we employ (CUT): for all $\langle a,b \rangle \in \mathsf{con}$ with $a \in F_+$ we have $v_+ \sqcup a \in \mathcal{S}_+$ by maximality of v_+ and so $\langle v_+ \sqcup a, v_- \rangle \in \mathsf{tot}$ by (2); if it was the case that $\langle v_+, v_- \rangle = \langle v_+, u_- \sqcup \bigsqcup^{\uparrow}\mathsf{P}(v_+) \rangle \in \mathsf{tot}$, then $\langle v_+, u_- \rangle \in \mathsf{tot}$ would follow, contradicting (hm$_{\mathsf{tot}}$).

For part (6) let $y \in \mathcal{U}_-$. By regularity and the assumption that $\mathcal{U}_-$ is completely prime, some $y' \lhd y$ also belongs to $\mathcal{U}_-$. The witness $r_{y' \lhd y}$ is not in $\mathcal{S}_+$ because of $\langle r_{y' \lhd y}, y' \rangle \in \mathsf{con}$ and assumption (hm$_{\mathsf{con}}$). By part (5) there is a point (F_+, F_-) that separates $r_{y' \lhd y}$ from $\mathcal{S}_+$. By (4) we have that $F_- \subseteq \mathcal{U}_-$ and because of $\langle r_{y' \lhd y}, y \rangle \in \mathsf{tot}$ it must also be the case that $y \in F_-$.

Finally, consider the last claim; the two descriptions of A are equal because of (4). Any $\mathcal{T}_+$-open neighbourhood of A has the form $\Phi_+(x)$ with $x \in \mathcal{S}_+$ by (5). It follows that A is $\mathcal{T}_+$-compact. Only the maximality of u_- in $L_- \setminus \mathcal{U}_-$ is required to see that A is the complement of $\Phi_-(u_-)$. ∎

Note that the infinitary rule (CUT) and the Axiom of Choice are only required to establish the link between the d-frame and its spectrum.

Theorem 5.5 *In any d-frame $\mathcal{L}$ there is a one-to-one correspondence between*

(i) pairs $(\mathcal{S}_+, \mathcal{U}_-)$ satisfying $(\mathrm{hm}_{\mathsf{con}})$ *and* $(\mathrm{hm}_{\mathsf{tot}})$*, and*

(ii) maps q from $\mathcal{L}$ to the four-element d-frame 2.2 *which preserve tt, $\bigsqcup^{\uparrow}$,* con*,* tot*, and the logical operation $\wedge$.*

If furthermore the d-frame is regular and satisfies (CUT)*, then these are in one-to-one correspondence with*

(iii) subsets K of $\mathsf{spec}\,\mathcal{L}$ *which are compact saturated in the positive and closed in the negative topology.*

Proof. Given a map q as described in part (ii), consider $\mathcal{S}_+ = q^{-1}(\{tt, 1\}) \cap L_+$ and $\mathcal{U}_- = q^{-1}(\{ff, 1\}) \cap L_-$. It is straightforward to show that the pair $(\mathcal{S}_+, \mathcal{U}_-)$ satisfies $(\mathrm{hm}_{\mathsf{con}})$ and $(\mathrm{hm}_{\mathsf{tot}})$. For the translation in the opposite direction let $q(\alpha) = \bigsqcup(\{tt \mid \alpha_+ \in \mathcal{S}_+\} \cup \{ff \mid \alpha \in \mathcal{U}_-\})$.

The translation from (i) to (iii) was given in the preceding lemma; for the reverse let $K \subseteq \mathsf{spec}\,\mathcal{L}$ as described in (iii) and set $\mathcal{S}_+ = \{x \mid K \subseteq \Phi_+(x)\}$ and $\mathcal{U}_- = \{y \mid K \cap \Phi_-(y) \neq \emptyset\}$. Showing that these translations are inverses of each other requires nothing more than an unwinding of the definitions. ∎

In reference to this result, we call the pair $(\mathcal{S}_+, \mathcal{U}_-)$ an *HM-pair*, the corresponding map q an *HM-map*, and (if applicable) the corresponding set K an *HM-set*.

Let us discuss Theorem 5.5 in terms of HM-maps. Given a *consistent* predicate φ, that is, $\varphi \in \mathsf{con}$, the value of q at φ can only be tt, ff, or 0. The first outcome indicates that *all* elements of K satisfy φ, the second that *some* element of K fails φ, and the last that neither holds (which is a possibility because a consistent predicate does not need to be Boolean). This means that HM-maps act like *universal quantifiers* for (consistent) predicates: $q \colon \varphi \mapsto \forall x \in K.\, \varphi(x)$.

Generally, one would expect a universal quantifier to preserve tt (because $\forall x \in K.\, tt$ is valid); on the other hand, $\forall x \in K.\, ff$ is false only if K is non-empty, so q need not preserve ff. Also, one would expect it to preserve conjunction ($\wedge$) but not disjunction ($\vee$), and certainly one would not want it to be inconsistent (returning 1) for a consistent predicate, or to be undecided (returning 0) for a total predicate, that is, one expects it to preserve con and tot.

The preservation of $\bigsqcup^{\uparrow}$ can be seen as a *computability* condition on the universal quantifier: If a (consistent) predicate φ is the directed supremum of (consistent) predicates φ_i, and if the universal quantifier applied to φ returns a definite answer, that is, either tt or ff, then computability requires the same answer be obtained from an approximant φ_i already.

All in all, then, Theorem 5.5 is a generalisation of the theory of continuous quantification on topological spaces, discovered by Taylor [23] and Escardó [8], to a logic in which predicates are allowed to have value ff as well as tt.

For a version of Theorem 5.5 on the side of bitopological spaces we first observe that regularity implies that the space is order-separated, so by Theorem 2.9 it is automatically d-sober. In an order-separated space a τ_+-compact saturated set is also τ_--closed. Furthermore, the corresponding d-frame ΩX satisfies (CUT) by Proposition 4.5, and so 5.5 applies:

Theorem 5.6 *If $(X; \tau_+, \tau_-)$ is a regular bitopological space then there is a one-to-one correspondence between*

(i) maps from ΩX to 2.2 which preserve tt, $\bigsqcup^{\uparrow}$, con, tot and $\wedge$, and

(ii) subsets K of X which are compact saturated with respect to τ_+.

6 Existential quantification and compactness

The construction of the previous section also provides us with a notion of an *existential* quantifier for four-valued predicates. The idea is to adapt the classical translation $\exists x.\varphi \leftrightarrow \neg\forall x.\neg\varphi$ to the present setting. This is achieved by reversing the roles of L_+ and L_-, and similarly by exchanging tt and ff in 2.2. So we consider pairs $(\mathcal{U}_+, \mathcal{S}_-)$ consisting of a completely prime upper set in L_+ and a Scott-open filter in L_- satisfying the analogues of $(\mathrm{hm}_{\mathsf{con}})$ and $(\mathrm{hm}_{\mathsf{tot}})$. The corresponding quantifier q maps a predicate $\varphi = \langle \varphi_+, \varphi_- \rangle$ to tt if $\varphi_+ \in \mathcal{U}_+$, to ff if $\varphi_- \in \mathcal{S}_-$, and to 1 if both hold. This can be written as $q(\varphi) = \bigsqcup(\{tt \mid \varphi_+ \in \mathcal{U}_+\} \cup \{ff \mid \varphi_- \in \mathcal{S}_-\})$ as we did for universal quantification before. One now checks without difficulties that q preserves ff, $\bigsqcup^{\uparrow}$, con, tot, and $\vee$, that is, it behaves like a *continuous existential quantifier*. If the d-frame $\mathcal{L}$ is regular and satisfies the analogue of (CUT) (with the roles of L_+ and L_- exchanged) then q corresponds to a uniquely determined set G of $\mathsf{spec}\,\mathcal{L}$ that is closed with respect to τ_+ and compact saturated with respect to τ_-. In other words, we can interpret $q(\varphi)$ as $\exists x \in G.\, \varphi(x)$.

For $(X; \tau_+, \tau_-)$ a regular bitopological space, Theorem 5.6 gives us a complete overview of the sets which admit universal, respectively, existential quantification. To repeat, the former are exactly the τ_+-compact saturated ones while the latter are those subsets that are compact saturated with respect to τ_-. Since a regular bitopological space is order separated, the specialisation order $\leq_+$ equals $\geq_-$ (as shown in Lemma 2.8) and it follows that universally quantifiable sets are upwards closed whereas existentially quantifiable ones are downward closed. Consequently, it is rare for a set to have both qualities.

In the absence of spatiality we consider quantifiers rather than quantifiable subsets. To begin with, we have the following observation:

Proposition 6.1 *Let $\mathcal{L}$ be a reasonable d-frame. Then $\varepsilon_\forall \colon \mathcal{L} \to 2.2$, $\varepsilon_\forall(\varphi) = tt$ for all $\varphi \in \mathcal{L}$, is a continuous universal quantifier. Its associated HM-pair is $(L_+, \emptyset)$.*

If q, q' are continuous universal quantifiers then $q \wedge q'$, defined by $q \wedge q'(\varphi) := q(\varphi) \wedge q'(\varphi)$ is also one. If $(\mathcal{S}_+, \mathcal{U}_-)$ and $(\mathcal{S}'_+, \mathcal{U}'_-)$ are the HM-pairs associated with q and q', respectively, then $(\mathcal{S}_+ \cap \mathcal{S}'_+, \mathcal{U}_- \cup \mathcal{U}'_-)$ is the HM-pair associated with $q \wedge q'$.

It follows that the set of continuous universal quantifiers $\mathsf{univ}(\mathcal{L})$ carries the structure of a unital semilattice. An analogous result holds for the set $\mathsf{exist}(\mathcal{L})$ of continuous existential quantifiers, where the unit is $\varepsilon_\exists \colon \varphi \mapsto ff$ and the binary operation is given by $q \vee q'(\varphi) := q(\varphi) \vee q'(\varphi)$. Next, recall that a completely prime set $\mathcal{U}$ in a complete lattice L can alternately be described by $u = \bigsqcup(L \setminus \mathcal{U})$, the largest element not in $\mathcal{U}$. The preceding proposition then yields:

Proposition 6.2 *The assignment $A \colon q \mapsto \bigsqcup(L_- \setminus \mathcal{U}_-)$, where $(\mathcal{S}_+, \mathcal{U}_-)$ is the HM-pair associated with q, is a unital semilattice homomorphism from $(\mathsf{univ}(\mathcal{L}); \wedge, \varepsilon_\forall)$ to $(L_-; \sqcap, 1)$. Likewise, $E \colon q \mapsto \bigsqcup(L_+ \setminus \mathcal{U}_+)$ is a homomorphism from $(\mathsf{exist}(\mathcal{L}); \vee, \varepsilon_{\mathsf{exist}})$ to $(L_+; \sqcap, 1)$.*

(Note that E reverses the natural order associated with $\vee$ on $\mathsf{exist}(\mathcal{L})$.)

For a map in the reverse direction of A let $u_- \in L_-$ and consider

$$\begin{array}{rcl} \mathcal{U}_-(u_-) & := & L_- \setminus \downarrow u_- \\ \mathcal{S}_+(u_-) & := & \{x \in L_+ \mid \langle x, u_- \rangle \in \mathsf{tot}\} \end{array}$$

Clearly, $\mathcal{U}_-(u_-)$ is a completely prime upper set in L_-, and as long as $\mathcal{L}$ is reasonable, $\mathcal{S}_+(u_-)$ is a filter. Furthermore, the conditions ($\mathrm{hm}_{\mathsf{tot}}$) (by construction) and ($\mathrm{hm}_{\mathsf{con}}$) (because of ($\mathsf{con}-\mathsf{tot}$)) are satisfied. The only property missing is Scott-openness of $\mathcal{S}_+(u_-)$, or equivalently, Scott-continuity of the associated quantifier. This motivates the following definition:

Definition 6.3 *Say that a reasonable d-frame $\mathcal{L}$* supports continuous quantification *if for every $u_- \in L_-$, the set $\mathcal{S}_+(u_-) = \{x \in L_+ \mid \langle x, u_- \rangle \in \mathsf{tot}\}$ is Scott-open, and the same is true for $\mathcal{S}_-(u_+) = \{y \in L_- \mid \langle u_+, y \rangle \in \mathsf{tot}\}$ for every $u_+ \in L_+$.*

Proposition 6.4 *If $\mathcal{L}$ supports continuous quantification then the assignment*

$$a\colon u_- \mapsto q_{u_-}$$

where q_{u_-} is the universal quantifier associated with the HM-pair $(\mathcal{S}_+(u_-), \mathcal{U}_-(u_-))$ is a unital semilattice homomorphism from $(L_-; \sqcap, 1)$ to $(\mathsf{univ}(\mathcal{L}); \wedge, \varepsilon_\forall)$. When $\mathsf{univ}(\mathcal{L})$ is equipped with the natural order derived from the semilattice operation then one has $a \circ A \leq \mathsf{id}_{\mathsf{univ}(\mathcal{L})}$ and $A \circ a = \mathsf{id}_{L_-}$, that is, a is adjoint to the map A of Proposition 6.2.

Likewise, the assignment

$$e\colon u_+ \mapsto q_{u_+}$$

where q_{u_+} is the existential quantifier associated with the HM-pair $(\mathcal{U}_+(u_+), \mathcal{S}_-(u_+))$ is a homomorphism from $(L_+; \sqcap, 1)$ to $(\mathsf{exist}(\mathcal{L}); \vee, \varepsilon_\exists)$. Like the map E of Proposition 6.2 it is order-reversing when $\mathsf{exist}(\mathcal{L})$ is equipped with the natural order, and one has $e \circ E \leq \mathsf{id}_{\mathsf{exist}(\mathcal{L})}$ and $E \circ e = \mathsf{id}_{L_+}$.

If $\mathcal{L}$ is also regular, then A and E are isomorphisms with inverses a and e, respectively.

Proof. We check that the map a is a homomorphism. Regarding the unit, this boils down to showing that $\mathcal{S}_+(1) = \{x \in L_+ \mid \langle x, 1 \rangle \in \mathsf{tot}\}$ equals L_+, which is indeed true because ($\mathsf{tot}-\mathit{ff}$) and ($\mathsf{tot}-\uparrow$) hold in a reasonable d-frame. To show that a preserves the semilattice operation, we must compare

$\mathcal{S}_+^1 := \{x \in L_+ \mid \langle x, u_- \sqcap u'_- \rangle \in \mathsf{tot}\}$ with $\mathcal{S}_+^2 := \{x \in L_+ \mid \langle x, u_- \rangle \in \mathsf{tot}\} \cap \{x \in L_+ \mid \langle x, u'_- \rangle \in \mathsf{tot}\}$. We have $\mathcal{S}_+^1 \subseteq \mathcal{S}_+^2$ because of ($\mathsf{tot}-\uparrow$) and $\mathcal{S}_+^2 \subseteq \mathcal{S}_+^1$ because of ($\mathsf{tot}-\vee$). The equality $L_- \setminus \downarrow(u_- \sqcap u'_-) = (L_- \setminus \downarrow u_-) \cup (L_- \setminus \downarrow u'_-)$ is trivial.

For the adjointness property observe that $\mathcal{S}_+(u_-)$ is indeed the smallest set that together with $\mathcal{U}_-(u_-)$ satisfies (hm_{tot}), which shows $a \circ A \leq \mathsf{id}_{\mathsf{univ}(\mathcal{L})}$; the equality $A \circ a = \mathsf{id}_{L_-}$ is trivial.

Finally, in Lemma 5.4(2) it was shown that in the presence of regularity at most one Scott-open filter can be paired with a given completely prime upper set $\mathcal{U}$; this proves that $a \circ A = \mathsf{id}_{\mathsf{univ}(\mathcal{L})}$ holds in this case. ∎

As it turns out, there is a simple and topologically meaningful characterisation of the d-frames that support continuous quantification.

Definition 6.5 *A reasonable d-frame is called* compact *if* tot*, viewed as a subset of* $L_+ \times L_-$*, is Scott-open.*

For bitopological spaces compactness means that any cover with open sets from *both* topologies has a finite subcover.

Theorem 6.6 *A d-frame* $\mathcal{L}$ *supports continuous quantification if and only if it is compact.*

Proof. "if" Since tot is a Scott-open set, the filter $\mathcal{S}_+(u_-) = \{x \in L_+ \mid \langle x, u_- \rangle \in \mathsf{tot}\}$ is clearly Scott-open as well.

"only if" In a product lattice, Scott-openness can be checked in each coordinate separately, [2, Lemma 3.2.6], so assume $(x_i)_{i \in I}$ is a directed set in L_+ such that $\langle \bigsqcup^{\uparrow} x_i, y \rangle \in \mathsf{tot}$. This is tantamount to saying that $\bigsqcup^{\uparrow} x_i \in \mathcal{S}_+(y)$ and since it is assumed that the latter is Scott-open, some x_{i_0} will belong to $\mathcal{S}_+(y)$ already. In other words, $\langle x_{i_0}, y \rangle \in \mathsf{tot}$. ∎

Somewhat to our surprise, the preceding statement does not rely on regularity, though it has to be said that in the absence of regularity the intuitions about quantification, as developed in the previous section, are not valid. This is because in a non-regular d-frame (or a non-regular bitopological space) the connection between the two frames (resp., topologies) is very loose. For example, a τ_--closed set need not even be τ_+-saturated, etc. In contrast, compact *regular* d-frames are extremely well-behaved and play a central role in the construction and analysis of semantic spaces. There is no room here to expand on this connection; instead we refer the interested reader to [16], sections 6 and 8.1, and the references given there. For the present paper we combine Proposition 6.4 and Theorem 6.6 to conclude that for compact regular d-frames there is an isomorphism between the elements of L_- and continuous universal quantifiers, and between the elements of L_+ and continuous existential quantifiers.

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