The probabilistic powerdomain for stably compact spaces

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September 22, 2011

Abstract

This paper reviews the one-to-one correspondence between stably compact spaces (a topological concept covering most classes of semantic domains) and compact ordered Hausdorff spaces. The correspondence is extended to certain classes of real-valued functions on these spaces. This is the basis for transferring methods and results from functional analysis to the non-Hausdorff setting.

As an application of this, the Riesz Representation Theorem is used for a straightforward proof of the (known) fact that every valuation on a stably compact space extends uniquely to a Radon measure on the Borel algebra of the corresponding compact Hausdorff space.

The view of valuations and measures as certain linear functionals on function spaces suggests considering a weak topology for the space of all valuations. If these are restricted to the probabilistic or sub-probabilistic case, then another stably compact space is obtained. The corresponding compact ordered space can be viewed as the set of (probability or sub-probability) measures together with *their* natural weak topology.

1 Introduction

[Note: This paper was published as

> M. Alvarez-Manilla, A. Jung, and K. Keimel. The probabilistic powerdomain for stably compact spaces. *Theoretical Computer Science*, 328:221–244, 2004

As pointed out to us by Gayle Apfel, the statement of Proposition 13 was wrong. This is corrected in this version.]

In denotational semantics programs and program fragments are mapped to elements of mathematical structures, such as "domains" in the sense of Scott, [Sco70, Sco82]. If the system to be modelled has the ability to make random (or pseudo-random) choices, then it makes sense to model its behaviour by a measure which records the probability for the system to end up in a measurable subset of

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the set of possible states. These ideas were first put forward by Saheb-Djahromi, [SD80], and Kozen, [Koz81]. The former considered (probability) measures on the Borel-algebra generated by Scott-open sets of a dcpo, while the latter worked with abstract measure spaces.

From a computational point of view it makes sense to measure only *observable* subsets of the state space. These, in turn, can often be identified with the *open* sets of a natural topology, for example, the Scott topology on domains. This connection between computability and topology was most clearly expounded by Smyth, [Smy83, Smy92], and the idea was then carried further by Abramsky, [Abr91], Vickers, [Vic89], and others.

A function $\mu\colon \mathcal{G} \to \mathbb{R}_+$ which assigns a "weight" to the open sets of a topological space $(X, \mathcal{G})$ is called a *valuation* if it satisfies the axioms

$$\begin{array}{rcl} \mu(\emptyset) & = & 0 \\ \forall U, V \in \mathcal{G}.\ U \subseteq V & \Rightarrow & \mu(U) \leq \mu(V) \\ \forall U, V \in \mathcal{G}.\ \mu(U) + \mu(V) & = & \mu(U \cup V) + \mu(U \cap V) \end{array}$$

A *probability valuation* is obtained when $\mu(X) = 1$ holds. This notion first arose within Mathematics, [Bir67, HT48, Pet51], and while one could say that within Computer Science it was implicit in the aforementioned [SD80], it was only explicitly adopted in [JP89] by Jones and Plotkin.

Comparing this work with the earlier approach by Saheb-Djahromi or Kozen it is natural to ask whether valuations can be extended to Borel measures, or whether the latter are intrinsically more informative than the former. As has been established by a number of authors, e.g. [Law82, AMESD00, AM01], and with a number of techniques, *continuous* valuations do indeed uniquely extend to measures on large classes of spaces. The present paper adds another proof of this important fact in the case of *stably compact spaces*.

Why another proof? We believe that our approach has a number of attractive features, not least of which are its brevity and simple structure. In essence, we study valuations and measures through their effect on (continuous) functions via integration, and achieve the actual extension by invoking the Riesz Representation Theorem. Continuous functions, of course, are central to Analysis but they have also appeared in denotational semantics literature: [Jon90, Chapters 6 and 7] uses them to establish a duality as a basis for a program logics; [DGJP99] view them as "tests" on a labelled Markov system.

The route via functions is also useful for the second concern of this paper, namely, the question of constructing a semantic domain from the set of valuations on a domain. We mentioned already Saheb-Djahromi's observation that valuations carry a natural order which turns them into dcpos. Jones extends this to the (technically difficult) result that continuity (in the sense of "continuous domain") is also preserved. Unfortunately, a further strengthening of this has not yet been possible, that is to say, we do not know whether the valuations on an FS-domain ([Jun90]) or a retract of SFP form another such structure; [JT98] points out errors in published work and summarises the partial results which have been obtained to date.

The approach taken here is somewhat different from this work. Instead of working with the order between valuations, we consider semantic domains as *topological spaces* and seek a natural topology on the set of valuations. There are a number of possibilities here, for example, the Scott topology arising from the dcpo-order. However, we take our cue from the representation of valuations as certain functionals on continuous real-valued functions and choose a *weak topology* in the sense of functional analysis. This is certainly consistent with earlier work as we know that the weak topology is the same as the Scott topology when one starts with a continuous domain, [Kir93, Satz 8.6], [Tix95, Satz 4.10]. The point here is to consider the weak topology in a situation where the order-relation is too sparse to sufficiently restrict the Scott topology. The natural setting for our results, then, is that of stably compact spaces. These subsume most semantic domains (such as “FS” or “SFP”) and have been shown to have many other closure properties of interest to semanticists, [Keg99]. Most relevant for the current discussion is the fact that they are in one-to-one correspondence to a simple program logic in the vein of Abramsky’s “Domain Theory in Logical Form”, [Abr91]. Indeed, the space of valuations in its weak topology can be characterised through a finitistic construction on the logical side, and the results presented here give further credibility to the axioms chosen in [MJ02].

Although of interest for some time to a core of researchers in semantics and Stone duality, stably compact spaces are not as widely known in Computer Science as they deserve. We take care, therefore, to develop their basic theory in an entirely elementary manner at the beginning of our paper. For this we choose a slightly different (though equivalent) axiomatisation which illustrates the slogan that *stably compact spaces are T_0-spaces in which compact sets behave in the same way as in the Hausdorff setting*.

We acknowledge with pleasure discussions on material in this paper with Martín Escardó, Reinhold Heckmann, Ralph Kopperman, and Jimmie Lawson.

This paper arose as an amalgamation and extension of [Jun04] and [Kei04].

2 Compact ordered and stably compact spaces

2.1 Compact ordered spaces

A *partially ordered topological space* (or *ordered space*, for short) in the sense of Nachbin [Nac65] is a set X with a topology $\mathcal{O}$ and a partial order $\leq$ such that the graph of the order is closed in $X \times X$. This captures the natural assumption that, for two converging nets $x_i \to x$ and $y_i \to y$, the property $x_i \leq y_i$ for all $i \in I$ implies $x \leq y$. In terms of open sets, this is equivalent to saying that for any two points $x \not\leq y$ in X there are open sets U containing x and V containing y such that for every $x' \in U$ and $y' \in V$, $x' \not\leq y'$ holds. It follows that ordered spaces are Hausdorff.

A subset U of X is called an *upper* (*lower*) set, if $x \in U$ implies $y \in U$ for all $y \geq x$ (resp., $y \leq x$). The smallest upper (lower) set containing a subset A is denoted $\uparrow A$ (resp., $\downarrow A$). In an ordered space sets of the form $\uparrow x = \uparrow\{x\}$ or $\downarrow x = \downarrow\{x\}$ are always closed, and more generally, this is true for $\uparrow A$ and $\downarrow A$ where A is compact. This little observation has strong consequences in case the ordered space is compact, as was first noted by Leopoldo Nachbin [Nac65]:

Lemma 1 ([Nac65])**.** *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space.*

(i) (Order normality) *Let A and B be disjoint closed subsets of X, where A is an upper and B is a lower set. Then there exist disjoint open neighbourhoods $U \supseteq A$ and $V \supseteq B$ where again U is an upper and V is a lower set.*

(ii) (Order separation) *Whenever $x \not\leq y$ there exist an open upper set U containing x and an open lower set V containing y which are disjoint.*

(iii) (Order Urysohn property) *For every pair A, B of disjoint closed subsets, where A is an upper and B is a lower set, there exists a continuous order-preserving function into the unit interval which has value 1 on A and 0 on B.*

Proof. By normality of compact Hausdorff spaces, A and B have disjoint open neighbourhoods U' and V'. Set $U = X \setminus \downarrow(X \setminus U')$ and $V = X \setminus \uparrow(X \setminus V')$. Order separation is a special case of order normality, and the order preserving version of Urysohn's Lemma follows, as usual, by repeated application of order normality. □

2.2 The upwards topology of a compact ordered space

One way to interpret this lemma is to say that there is an abundance of open *upper* sets in a compact ordered space. For any ordered space, the collection

$$\mathcal{U} := \{U \in \mathcal{O} \mid U = \uparrow U\}$$

of open upper sets is a topology coarser than the original one; we call it the *topology of convergence from below* or *upwards topology* for short. The resulting topological space $(X, \mathcal{U})$ we denote by $X^{\uparrow}$.

Sets of the form $X \setminus \downarrow x$ always belong to $\mathcal{U}$ and therefore every upper set is equal to the intersection of its $\mathcal{U}$-open neighbourhoods, that is, it is $\mathcal{U}$-*saturated*. The converse direction being trivial, we thus have:

Proposition 2. *In an ordered space the upper sets are precisely the $\mathcal{U}$-saturated ones.*

For a general topological space $(X, \mathcal{G})$ one sets $x \leq_{\mathcal{G}} y$ if every neighbourhood of x also contains y. This is always a preorder and it is anti-symmetric if and only if the space is T_0. It is called the *specialisation order associated with* $\mathcal{G}$. The preceding proposition tells us that $\leq_{\mathcal{U}}$ is precisely the original order $\leq$ in any ordered space.

In order to analyse the properties of $\mathcal{U}$ further in the case where $(X, \mathcal{O}, \leq)$ is compact, we also consider the set of compact saturated sets:

$$\mathcal{K}_{\mathcal{U}} := \{K \subset X \mid K \text{ is } \mathcal{U}\text{-saturated and } \mathcal{U}\text{-compact}\}$$

Lemma 3. *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space. The elements of $\mathcal{K}_{\mathcal{U}}$ are precisely those subsets of X which are upper and closed with respect to $\mathcal{O}$.*

Proof. The upper closed sets of X are $\mathcal{U}$-compact because the topology $\mathcal{U}$ is weaker than $\mathcal{O}$. For the converse one uses order separation. □

We now have enough information to show that from $\mathcal{U}$ alone we can reconstruct the original compact ordered space. In general, one considers the *patch topology* $\mathcal{G}_p$ of a topological space $(X, \mathcal{G})$ by augmenting $\mathcal{G}$ with complements of compact saturated sets. With this terminology we can formulate the following:

Theorem 4. *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space. Then $\mathcal{O} = \mathcal{U}_p$ and $\leq\, = \,\leq_{\mathcal{U}}$.*

Proof. Because of Lemma 3, $\mathcal{U}_p$ is contained in $\mathcal{O}$. It is Hausdorff because of order separation and therefore the identity map $i\colon (X, \mathcal{O}) \to (X, \mathcal{U}_p)$ is a homeomorphism.

The possibility to reconstruct the order out of the upwards topology has been remarked before. □

Since with $(X, \mathcal{O}, \leq)$, the "upside-down" space $(X, \mathcal{O}, \geq)$ is also compact ordered, the results in this section hold equally well for the *topology* $\mathcal{D}$ *of convergence from above* or *downwards topology*. By Lemma 3, its open sets are precisely the complements of the compact saturated sets of $\mathcal{U}$.

2.3 Stably compact spaces

As it turns out, topologies which arise as upwards topologies in compact ordered spaces can be characterised intrinsically. We begin with the following observations:

Proposition 5. *For a compact ordered space* $(X, \mathcal{O}, \leq)$ *the upwards topology* $\mathcal{U}$ *is*

(i) T_0*;*

(ii) compact;

(iii) locally compact;

(iv) coherent, that is, pairs of compact saturated sets have compact intersection;

(v) well-filtered, that is, for any filter base $(A_i)_{i \in I}$ *of compact saturated sets, for which* $\bigcap_i A_i$ *is contained in an open upper set* U*, there is an index* i_0 *such that* A_{i_0} *is contained in* U *already.*

Proof. The T_0 separation property follows from order separation, (ii) is trivially true because $\mathcal{U}$ is weaker than $\mathcal{O}$, and (iii) is a reformulation of order normality. Coherence and well-filteredness follow from Lemma 3. □

Definition 6. *A* T_0 *space which is compact, locally compact, coherent, and well-filtered is called* stably compact*.*

In recent literature it has been customary to use “sober” instead of “well-filtered” in the definition of stably compact spaces. However, in the presence of local compactness these two properties are equivalent, [GHK$^+$03, Theorem II-1.21]. With this note we would like to make a case for the revised definition, because it makes it apparent that *stably compact spaces are the* T_0*-analogue of compact Hausdorff spaces*, in the sense that compact saturated sets in the former have the same properties as compact subsets in the latter. The following lemma illustrates this:

Lemma 7. *Let* $(X, \mathcal{U})$ *be a stably compact space. Then any collection of compact saturated subsets has compact intersection.*

Proof. Finite intersections leading again to compact saturated subsets, we can assume the collection to be filtered. By well-filteredness, an open cover of the intersection will contain an element of the filter base already. This being compact, a finite subcover will suffice. □

This result justifies the following definition.

Definition 8. *Let* $(X, \mathcal{U})$ *be a stably compact space. The* co-compact topology $\mathcal{U}_\kappa$ *on* X *is given by the complements of compact saturated sets.*

If the stably compact space $(X, \mathcal{U})$ arose as the topology of convergence from below in a compact ordered space, then Lemma 3 implies that the co-compact topology derived from $\mathcal{U}$ is the same as the topology of convergence from above.

The following proposition is reminiscent of the well-known fact that a compact Hausdorff-topology cannot be weakened without losing separation.

Proposition 9. *Let $(X, \mathcal{U})$ be a stably compact space. Let further $\mathcal{B}$ be a subset of $\mathcal{U}$ and $\mathcal{C}$ a subset of the co-compact topology $\mathcal{U}_\kappa$, such that the following property holds:*

$$\forall x, y \in X.\ x \not\leq_{\mathcal{U}} y \implies \exists U \in \mathcal{B}, L \in \mathcal{C}.\ x \in U, y \in L, L \cap U = \emptyset\ .$$

Then $\mathcal{B}$ is a subbasis for $\mathcal{U}$.

Proof. Let x be an element of an open set $O \in \mathcal{U}$. Then by assumption for every y in $X \setminus O$ there exist disjoint sets $U_y \in \mathcal{B}$ and $L_y \in \mathcal{C}$ which contain x and y, respectively. The complements of the L_y are compact saturated by definition and their intersection is contained in O. Well-filteredness tells us that the same is true for a finite subcollection of L_y's. The intersection of the corresponding U_y is a neighbourhood of x contained in O. $\square$

Corollary 10. *Let $\mathcal{U}$ and $\mathcal{U}'$ be stably compact topologies on a set X such that $\leq_{\mathcal{U}} = \leq_{\mathcal{U}'}$, $\mathcal{U} \subseteq \mathcal{U}'$, and $\mathcal{U}_\kappa \subseteq \mathcal{U}'_\kappa$. Then $\mathcal{U} = \mathcal{U}'$.*

We are now ready to complete the link with compact ordered spaces.

Theorem 11. *Let $(X, \mathcal{U})$ be a stably compact space. Consider its patch topology $\mathcal{U}_p$ and specialisation order $\leq_{\mathcal{U}}$. Then $(X, \mathcal{U}_p, \leq_{\mathcal{U}})$ is a compact ordered space. Furthermore, the upwards topology arising from $\mathcal{U}_p$ and $\leq_{\mathcal{U}}$ is equal to $\mathcal{U}$, and the co-compact topology $\mathcal{U}_\kappa$ is equal to the topology of convergence from above derived from $\mathcal{U}_p$ and $\leq_{\mathcal{U}}$.*

Proof. The Hausdorff separation property and the closedness of $\leq_{\mathcal{U}}$ follow from T_0 and local compactness. Compactness of the patch topology requires the Axiom of Choice in the form of Alexander's Subbase Lemma: Let $\mathcal{B} \cup \mathcal{C}$ be a covering of X where the open sets in $\mathcal{B}$ are chosen from $\mathcal{U}$ and the ones in $\mathcal{C}$ are complements of compact saturated sets. The points not covered by the elements of $\mathcal{C}$ form a compact saturated set by Lemma 7 and must be covered by elements of $\mathcal{B}$. A finite subcollection $\mathcal{B}' \subseteq_{\text{fin}} \mathcal{B}$ will suffice for the purpose. By well-filteredness, then, a finite intersection of complements of elements of $\mathcal{C}$ will be contained in $\bigcup \mathcal{B}'$ already. This completes the selection of a finite subcover.

The same argument shows that every compact saturated set in $(X, \mathcal{U})$ is also compact in the patch topology.

The specialisation order that one derives from the topology of convergence from below on the space $(X, \mathcal{U}_p, \leq_{\mathcal{U}})$ is the same as $\leq_{\mathcal{U}}$ by Theorem 4.

We are therefore in the situation described by Corollary 10 and can conclude that no new open upper sets arise in the patch construction. Lemma 3, then, tells us that the closed upper sets in $(X, \mathcal{U}_p, \leq_{\mathcal{U}})$ are precisely the compact saturated sets of $\mathcal{U}$. Hence the co-compact topology with respect to $\mathcal{U}$ is equal to the topology of convergence from below on $(X, \mathcal{U}_p, \leq_{\mathcal{U}})$. $\square$

Corollary 12. *Let $(X, \mathcal{U})$ be a stably compact space.*

(i) *The co-compact topology $\mathcal{U}_\kappa$ is also stably compact.*

(ii) $(\mathcal{U}_\kappa)_\kappa = \mathcal{U}$

2.4 Examples

The prime example of an ordered space is given by the real line with the usual topology and the usual order. The upwards topology in this case consists of sets of the form $]r, \infty[$ (plus $\mathbb{R}$ and $\emptyset$, of course), and non-empty compact saturated sets associated to this, in turn, are the sets of the form $[r, \infty[$. We denote the real line with the upwards topology by $\mathbb{R}^{\uparrow}$. Also of interest to us is the non-negative part of this, denoted by $\mathbb{R}^{\uparrow}_{+}$. One obtains a *compact* ordered space by either restricting to a compact subset, such as the unit interval, or by extending the real line with elements at infinity in the usual way, denoted here by $\overline{\mathbb{R}} = [-\infty, \infty]$ and $\overline{\mathbb{R}}_{+} = [0, \infty]$.

In general, one cannot expect a compact ordered space to be fully determined by its order alone, after all, every compact Hausdorff space can be equipped with a trivial closed order, namely, the identity relation. Semantic domains, however, do provide examples where the order structure is rich enough to determine a non-trivial stably compact topology. We review the definitions: A *dcpo* (for *directed-complete partial order*) is an ordered set in which every directed subset has a supremum. The closed sets of the *Scott topology* σ_D of a dcpo D are those lower sets which are closed against formation of directed suprema. It follows that a function between dcpos is continuous with respect to the two Scott topologies if and only if it preserves the order and suprema of directed sets. In order to emphasise the dcpo context, such functions are usually called *Scott-continuous*.

The specialisation order associated with the Scott topology, which is always T_0, will give back the original order of the dcpo. An element x of a dcpo D is *way-below* an element y (written $x \ll y$) if whenever y is below the supremum of a directed set $A \subseteq D$, then x is below some element of A. A dcpo D is *continuous* or a *domain* if every element equals the directed supremum of its way-below approximants.

The Scott topology of a domain is always well-filtered, [Jun89, Lemma 4.12], and coherence can be characterised in an order-theoretic fashion as well, see [Jun89, Lemma 4.18], [GHK$^+$03, Proposition III-5.12]. As a special case, coherence holds in every continuous complete lattice (known as *continuous lattice* for short). Two examples are of interest here: The unit interval $[0, 1]$ (or $\overline{\mathbb{R}}$ or $\overline{\mathbb{R}}_{+}$) is a continuous lattice and the Scott topology is precisely the topology of convergence from below, discussed before. An element x of $[0, 1]$ is way-below y if $x = 0$ or $x < y$. The other class of examples is given by open set lattices of locally compact spaces. Here, the way-below relation is characterised by $U \ll V$ if and only if there exists a compact saturated set K such that $U \subseteq K \subseteq V$. Stably compact spaces qualify, and their open set lattices have the additional property (not true in general) that $U \ll V_1$ and $U \ll V_2$ implies $U \ll V_1 \cap V_2$.

More general domains with a coherent Scott topology have been considered in Theoretical Computer Science; we refer the interested reader to [AJ94, Section 4.2.3] and [GHK$^+$03, Section III-5].

2.5 Morphisms and constructions

Although theorems 4 and 11 suggest that we can switch freely between compact ordered and stably compact spaces, a difference between the two standpoints does become apparent when one considers the corresponding morphisms: neither is a continuous map between stably compact spaces necessarily patch continuous, nor is every patch continuous function continuous with respect to the original topologies. Indeed, it is the fact that T_0-continuous maps arise in applications to denotational semantics which motivates our interest in stably compact spaces.

Nevertheless, a connection between *subclasses* of continuous maps can be made. A continuous map $f\colon X \to X'$ between locally compact spaces is called *perfect* if the preimage $f^{-1}(K)$ of every compact saturated set $K \subseteq X'$ is compact in X.[1] The following is true:

Proposition 13. *For stably compact spaces $(X, \mathcal{U})$ and $(X', \mathcal{U}')$ a map $f\colon X \to X'$ is perfect, if and only if it is continuous with respect to the patch topologies on X and X' and monotone (i.e., order preserving) with respect to the specialisation orders.*

Note 1 (September 22, 2011)**.** *The published version said "locally compact" rather than "stably compact" which was an unintended error, pointed out to us by Gayle Apfel.*

In the remainder of this section we study some constructions on spaces and how they interact with the translations given in Theorems 4 and 11.

Proposition 14. *Arbitrary products of stably compact spaces are stably compact, and the product topology equals the upwards topology of the product of the corresponding compact ordered spaces.*

Proof. Let $(X_i, \mathcal{U}_i)_{i\in I}$ be any family of stably compact spaces and let $(X_i, \mathcal{O}_i, \leq_i)$ be the corresponding compact ordered spaces. We prove the second claim because it entails the first. By Tychonoff's Theorem the product $\mathcal{O}$ of the patch topologies $\mathcal{O}_i$ is again compact Hausdorff, and the shape of basic open sets in the product gives immediately that the coordinatewise order $\leq$ is closed. So $(\prod_{i\in I} X_i, \mathcal{O}, \leq)$ is a compact ordered space.

A basic open set from the product of the $\mathcal{U}_i$ is also open in $\mathcal{O}$. For the converse we employ Proposition 9, where the product of the $\mathcal{U}_i$ plays the role of $\mathcal{B}$ and the product of the respective co-compact topologies $(\mathcal{U}_i)_\kappa$ plays the role of $\mathcal{C}$ in the stably compact space derived from $(\prod_{i\in I} X_i, \mathcal{O}, \leq)$. The separation property is obviously satisfied because $x \not\leq y$ means $x_i \not\leq y_i$ for some index i. □

Subspaces are more interesting as they do not, in general, preserve any of the properties under consideration, except that the order remains closed. However, we have the following:

Proposition 15. *Let Y be a patch-closed subset of a stably compact space $(X, \mathcal{U})$. Then Y is stably compact when equipped with the subspace topology $\mathcal{U}{\restriction_Y}$, and $(\mathcal{U}{\restriction_Y})_p = \mathcal{U}_p{\restriction_Y}$.*

Proof. The subspace $(Y, \mathcal{U}_p{\restriction_Y}, \leq{\restriction_{Y\times Y}})$ is of course again a compact ordered space. If A is a closed lower set in Y, then its lower closure $\downarrow A$ in X is again closed as A is compact in X. This shows that the upper opens of $(Y, \mathcal{U}_p{\restriction_Y}, \leq{\restriction_{Y\times Y}})$ belong to $\mathcal{U}{\restriction_Y}$. The converse inclusion is trivial. □

The second case where we know something about the stable compactness of a subspace is related to continuous retractions. This fact is mentioned in [Law88] already but the proof uses a different characterisation of stable compactness.

Proposition 16. *Let Y be a continuous retract of a stably compact space X. Then Y is stably compact.*

Proof. Let $e\colon Y \to X$ be the section and $r\colon X \to Y$ the retraction map (both continuous). We check the defining properties for stable compactness. First of all, Y is a T_0-space because e is injective. The compactness of Y follows from the continuity of the (surjective) map r. If $x \in O \subseteq Y$, with O open in Y, then $r^{-1}(O)$ is an open neighbourhood of $e(x)$. Hence there is an open set U and a compact

[1] For more general spaces, perfectness requires an additional property, see [Hof84].

saturated set L in X such that $e(x) \in U \subseteq L \subseteq r^{-1}(O)$. The image of L under r is compact in Y, is contained in O, and contains the open set $e^{-1}(U)$ which contains x. This proves that Y is locally compact.

For stability, let K_1, K_2 be compact saturated sets in Y. We get that $e(K_1)$ and $e(K_2)$ are compact in X and hence $\uparrow e(K_i)$ is compact saturated in X. By the stability of X the intersection $(\uparrow e(K_1)) \cap (\uparrow e(K_2))$ is compact again. Its image under r is precisely $K_1 \cap K_2$; it is compact in Y by the continuity of r. Well-filteredness is shown in the same way. $\square$

Note that e does not need to be a perfect map in general, so the result is not subsumed by Proposition 15 already.[2]

2.6 Real-valued functions

For an ordered space $(X, \mathcal{G}, \leq)$ there are a number of possible function spaces into the reals that one might be interested in. Depending on which structure of the reals is taken into account, one can distinguish at least the following:

- the set $\mathrm{C}(X)$ of all continuous functions into the real line;
- the set $\mathrm{CM}(X)$ of all continuous *order-preserving* (i.e., *monotone increasing*) functions into the reals;
- the set $\mathrm{LSC}(X)$ of all real-valued functions on X which are continuous with respect to $\mathcal{G}$ and the topology of convergence from below on $\mathbb{R}$. We call these the *lower semicontinuous functions*; they are characterized by the property that $\{x \in X \mid g(x) > r\}$ is an open upper set in X for every $r \in \mathbb{R}$.

If in the above definitions $\mathbb{R}$ is replaced by the set of *non-negative* reals, then one obtains the function spaces $\mathrm{C}_+(X)$, $\mathrm{CM}_+(X)$, and $\mathrm{LSC}_+(X)$. In order to express the condition that all functions be bounded in $\mathbb{R}$ we use the notation $\mathrm{C}_b(X)$, $\mathrm{CM}_b(X)$, and $\mathrm{LSC}_b(X)$.

Our primary object of interest is the class of *compact* ordered spaces and in what follows the most prominent function spaces will be $\mathrm{C}(X)$, $\mathrm{CM}_+(X)$, and $\mathrm{LSC}_{+,b}(X^\uparrow)$. Note that because of compactness, the functions in $\mathrm{C}(X)$ and $\mathrm{CM}_+(X)$ are automatically bounded, whereas for $\mathrm{LSC}_+(X^\uparrow)$ this need not be the case; our preference for $\mathrm{LSC}_{+,b}(X^\uparrow)$ is primarily to avoid unnecessary complication stemming from arithmetic with ∞.

From Proposition 13 it is clear that for a compact ordered space X, $\mathrm{CM}_+(X)$ is a subset of $\mathrm{LSC}_{+,b}(X^\uparrow)$, consisting of all perfect maps from $X^\uparrow$ to $\mathbb{R}_+^\uparrow$. The sets $\mathrm{CM}_+(X)$, $\mathrm{LSC}_{+,b}(X^\uparrow)$, and $\mathrm{LSC}_+(X^\uparrow)$ are positive cones, that is, they are closed under addition and scalar multiplication with non-negative real numbers. Furthermore, these cones are ordered in the obvious (i.e., pointwise) way. The set $\mathrm{C}(X)$, on the other hand, is an ordered vector space. The smallest subvector space generated by $\mathrm{CM}_+(X)$ inside $\mathrm{C}(X)$ consists of differences $f - g$ with $f, g \in \mathrm{CM}_+(X)$; we denote it by $(\mathrm{CM}_+ - \mathrm{CM}_+)(X)$. The following picture may help to visualise the containment relations between

[2] Perfectness of e is guaranteed if e is an upper adjoint. This situation is called an *insertion-closure pair* in [AJ94, Section 3.1.5].

these function spaces:

$$
\begin{array}{ccccc}
\mathrm{C}(X) & & & & \\
& \nwarrow & & & \\
& & (\mathrm{CM}_+ - \mathrm{CM}_+)(X) & & \mathrm{LSC}_{+,b}(X^\uparrow) \\
& & & \nwarrow \quad \nearrow & \\
& & & \mathrm{CM}_+(X) &
\end{array}
$$

For any $r \in \mathbb{R}$ we adopt the following notation for a function $g\colon X \to \mathbb{R}$:

$$[g > r] := \{x \in X \mid g(x) > r\} = g^{-1}(]r, +\infty[)\,.$$

We have the following approximation results:

Lemma 17 ([Edw78])**.** *Every element of $f \in \mathrm{LSC}_+(X^\uparrow)$ is the (pointwise) supremum of elements of $\mathrm{CM}_+(X)$.*

Proof. Note that $\mathrm{CM}_+(X)$ is closed under taking pointwise maximum, so the collection of approximants to $f \in \mathrm{LSC}_+(X^\uparrow)$ is certainly directed. For $x \in X$ and $r < f(x)$, consider $[f > r]$ which is an upper open set in X containing x. By the order Urysohn property (Lemma 1(iii)) we obtain a continuous monotone increasing function g which takes value 1 on $\uparrow x$ and 0 on $X \setminus [f > r]$, so $r \cdot g$ is an element of $\mathrm{CM}_+(X)$ below f which approximates f at point x up to "precision" r. □

Lemma 18. *Every element g of $\mathrm{LSC}_+(X)$ can be represented as a directed supremum of simple functions belonging to $\mathrm{LSC}_{+,b}(X)$ in the following way*

$$g = \sup_{n \in \mathbb{N}} \sum_{i=1}^{n2^n} \frac{1}{2^n} \chi_{[g > \frac{i}{2^n}]}$$

The proof is immediate from the definition of lower semicontinuity.

To approximate continuous functions, we consider $\mathrm{C}(X)$ as a Banach space with the sup-norm $\|f\|$. As we remarked before, the set $\mathrm{CM}_+(X)$ of all non-negative monotone increasing continuous real-valued functions is a cone in $\mathrm{C}(X)$. Furthermore, it is closed under products and contains the constant function $\mathbf{1}$.

Lemma 19. *([Edw78]) For a compact ordered space X, the vector space $(\mathrm{CM}_+ - \mathrm{CM}_+)(X)$ generated by the cone $\mathrm{CM}_+(X)$ is dense in $\mathrm{C}(X)$ with respect to the sup norm.*

Proof. From the remark preceding this lemma it follows that $(\mathrm{CM}_+ - \mathrm{CM}_+)(X)$ is a subalgebra of $\mathrm{C}(X)$ which contains the constant function $\mathbf{1}$. By the order Urysohn property it follows that for any elements $x \not\leq y$ in X, there is a function $f \in \mathrm{CM}_+(X)$ such that $f(x) = 1$ and $f(y) = 0$. Hence, $\mathrm{CM}_+(X)$ and, *a fortiori*, $(\mathrm{CM}_+ - \mathrm{CM}_+)(X)$ separate the points of X. The lemma now follows from the Stone-Weierstraß Theorem. □

3 Measures and valuations

3.1 Measures and positive linear functionals on $\mathrm{C}(X)$

Let X be any Hausdorff space and $\mathcal{B}$ the $\sigma-$algebra of Borel sets, that is, the $\sigma-$algebra generated by the open subsets of X. Recall that a Borel measure on X is a function $m\colon \mathcal{B} \to \overline{\mathbb{R}}$ such that

m is strict:	$m(\emptyset)$	$=$	$0\,,$
m is additive:	$m(A)+m(B)$	$=$	$m(A\cup B)\,,$ whenever $A,B\in\mathcal{B}(X)$ are disjoint ,
m is σ-continuous:	$m(\bigcup_{n\in\mathbb{N}} A_n)$	$=$	$\sup_{n\in\mathbb{N}} m(A_n)$ for every increasing sequence $(A_n)_{n\in\mathbb{N}}\in\mathcal{B}$.

It follows from strictness and σ-continuity that measures can only take non-negative values. A measure is called *inner regular*, if

$$m(A) = \sup\ \{m(K) \mid K\subseteq A \text{ and } K \text{ compact}\} \text{ for all Borel sets } A\,.$$

We say that m is a *Radon measure*[3], if it is inner regular and if $m(K) < +\infty$ for every compact subset K. For a bounded Radon measure, that is, a Radon measure such that $m(X) < +\infty$, inner regularity implies *outer regularity* by passing to complements:

$$m(A) = \inf\ \{m(U) \mid A\subseteq U \text{ and } U \text{ open}\} \text{ for all Borel sets } A\,.$$

We denote by

$\mathfrak{M}(X)$ the set of all bounded Radon measures on X, by
$\mathfrak{M}_{\leq 1}(X)$ the subset of all Radon measures with $m(X)\leq 1$, and by
$\mathfrak{M}_1(X)$ the set of Radon probability measures, i.e., $m(X)=1$.

On compact Hausdorff spaces all Borel measures are automatically regular, so in this case the qualifier "Radon" only expresses boundedness.

$\mathfrak{M}(X)$ is a cone in the vector space of all functions from $\mathcal{B}$ to $\mathbb{R}$, that is, the sum m_1+m_2 of two bounded Radon measures, and also the scalar multiple rm for any non-negative real number r, are again bounded Radon measures. The subsets $\mathfrak{M}_{\leq 1}(X)$ and $\mathfrak{M}_1(X)$ are convex. On $\mathfrak{M}(X)$ there is a natural order relation

$$m_1 \leq m_2 \ :\Longleftrightarrow\ m_1(A)\leq m_2(A) \text{ for all Borel sets } A\,.$$

This order is trivial for probability measures. More interesting for us is the so-called *stochastic preorder*, which we can define when X is an ordered space. It is given by the following formula:

$$m_1 \preccurlyeq m_2 \ :\Longleftrightarrow\ m_1(U)\leq m_2(U) \text{ for all open upper sets } U\,.$$

Here the word "preorder" highlights the fact that there is no guarantee that $\preccurlyeq$ is antisymmetric in general.[4]

Integration of functions can be a subtle affair when one allows measurable sets of measure ∞, unbounded functions, functions whose support is not compact, or non-continuous functions. Since

[3] For compact Hausdorff spaces, the term *regular Borel measure* is more commonly used than that of a *Radon measure*.
[4] The notion of a *stochastic order* has been introduced much earlier for probability measures (see e.g. [Edw78]).

we are interested in compact ordered spaces, bounded Radon measures and functions with continuity properties, none of these complications arise; one can define the the integral of a continuous function $f\colon X \to \mathbb{R}_+$ in any of the available frameworks. The following definition is particularly convenient for our purposes. We set

$$\int f\, dm := \int_0^{+\infty} m([f > r])\, dr\,,$$

where the integral on the right is obtained by ordinary Riemann integration. This is a Choquet-type definition of the integral (see [Cho53, p. 265], [Kön97, Section 11]). Let us explain why this definition makes sense: For every r, the set $[f > r]$ is open and has a measure $m\big([f > r]\big) \in \mathbb{R}_+$. The function $r \mapsto m\big([f > r]\big)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is monotone decreasing and $m\big([f > r]\big) = 0$ for $r \geq \|f\|$. Thus this function is Riemann integrable and the Riemann integral $\int_0^{+\infty} m([f > r])\, dr$, which is in fact an integral extended over the finite interval $[0, \|f\|]$, is a real number. One extends the definition to all continuous functions in the usual way.

The fundamental properties of integration can now be derived from the properties of the Riemann integral:

(i) *(Linearity)* For $r, s \in \mathbb{R}$ and $f, g \in \mathrm{C}(X)$, $\int (rf + sg)\, dm = r \int f\, dm + s \int g\, dm$.

(ii) *(Positivity)* For $f \in \mathrm{C}_+(X)$, $\int f\, dm \geq 0$ holds.

This says that for every Radon measure m on a compact Hausdorff space X, the map $f \mapsto \int f\, dm$ is a positive linear functional on $\mathrm{C}(X)$.

The famous Riesz Representation Theorem states that linearity and positivity completely characterise integration:

Theorem 20. *Let X be a compact Hausdorff space. Then for every positive linear functional φ on $\mathrm{C}(X)$ there is a unique Radon measure m such that*

$$\varphi(f) = \int f\, dm \quad \textit{for every } f \in \mathrm{C}(X)\,.$$

We denote with $\mathrm{C}^\dagger(X)$ the set of all positive linear functionals on the ordered vector space $\mathrm{C}(X)$. It is standard knowledge that this is a subcone of the vector space $\mathrm{C}^*(X)$ of all bounded linear functionals. It can be ordered by setting

$$\varphi \leq \psi \;:\Longleftrightarrow\; \forall f \in \mathrm{C}_+(X).\ \varphi(f) \leq \psi(f)\,.$$

As with measures, for compact ordered spaces X, a preorder will be of interest to us:

$$\varphi \preccurlyeq \psi \;:\Longleftrightarrow\; \forall f \in \mathrm{CM}_+(X).\ \varphi(f) \leq \psi(f)\,.$$

From the Riesz Representation Theorem it follows that the cones $\mathfrak{M}(X)$ and $\mathrm{C}^\dagger(X)$ are isomorphic, as integration is indeed linear in its measure argument. We can strengthen this by also taking the preorders into account:

Theorem 21. *For a compact ordered space $(X, \mathcal{O}, \leq)$ the preordered cones $(\mathfrak{M}(X), \preccurlyeq)$ and $(\mathrm{C}^\dagger(X), \preccurlyeq)$ are isomorphic.*

Proof. If $m \not\preccurlyeq m'$ there exists an open upper set U for which $m(U) > m'(U)$. By regularity, we find a compact saturated set K inside U for which $m(K) > m'(U)$. The order Urysohn property provides us with a continuous monotone increasing function f which takes value 1 on K and 0 on $X \setminus U$. We then have

$$\int f\, dm \geq m(K) > m'(U) \geq \int f\, dm'$$

and we see that the integration functionals are not comparable with respect to $\preccurlyeq$ either.

For the converse let $m(U) \leq m'(U)$ for all $U \in \mathcal{U}$, and let $f \in \mathrm{CM}_+(X)$. Since $[f > r]$ is an upper open set for all $r \in \mathbb{R}$, we get $\int f\, dm \leq \int f\, dm'$ directly from our definition of integration. $\square$

We will show below that for a compact ordered space the stochastic preorder is in fact antisymmetric.

3.2 Valuations and Scott-continuous linear functionals on $\mathrm{LSC}_{+,b}(X)$

Let $(X, \mathcal{G})$ be a topological space, not necessarily Hausdorff. A *valuation* on $\mathcal{G}$ is a function $\mu\colon \mathcal{G} \to \overline{\mathbb{R}}$ with the following properties:

$$\begin{array}{lrcl}
\mu \text{ is strict:} & \mu(\emptyset) & = & 0\,, \\
\mu \text{ is modular:} & \mu(U) + \mu(V) & = & \mu(U \cup V) + \mu(U \cap V)\,, \\
\mu \text{ is monotone increasing:} & U \subseteq V & \Rightarrow & \mu(U) \subseteq \mu(V)\,.
\end{array}$$

A valuation is called *(Scott-) continuous,* if

$$\mu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \mu(U_i) \text{ for every directed family of open sets } U_i \in \mathcal{G}\,.$$

We denote by $\overline{\mathfrak{V}}(X)$ the set of all continuous valuations on $\mathcal{G}$. A natural order between valuations is given by

$$\mu \preccurlyeq \nu \;\; :\Longleftrightarrow \;\; \mu(U) \leq \nu(U) \text{ for all open } U \in \mathcal{G}\,,$$

which we again call the *stochastic order* in anticipation of a theorem which we will prove in the next section. With respect to this order, $\overline{\mathfrak{V}}(X)$ is *directed complete,* more precisely:

Lemma 22. *For every family $(\mu_i)_{i \in I}$ of continuous valuations on $\mathcal{G}$, which is directed for the stochastic order, the pointwise supremum $\mu(U) = \sup_i \mu_i(U)$ is again a continuous valuation on $\mathcal{G}$.*

For continuous valuations we also define an addition and a multiplication by non-negative scalars r by $(\mu + \nu)(U) = \mu(U) + \nu(U)$ and $(r\mu)(U) = r\mu(U)$, where we adopt the convention $0 \cdot (+\infty) = 0$ as usual in Measure Theory.

We denote by

$\mathfrak{V}(X)$ the set of all *bounded* continuous valuations, that is, $\mu(X) < +\infty$, by
$\mathfrak{V}_{\leq 1}(X)$ the subset of all *sub-probability* valuations, that is, $\mu(X) \leq 1$, and by
$\mathfrak{V}_1(X)$ the subset of all *probability* valuations, that is, $\mu(X) = 1$.

We note that $\mathfrak{V}(X)$ is a cone in the vector space of all functions from $\mathcal{G}$ to $\mathbb{R}$ and that $\mathfrak{V}_{\leq 1}(X)$ and $\mathfrak{V}_1(X)$ are convex subsets which are directed complete for the order $\preccurlyeq$.

In the same way that one can define the integral with respect to a Radon measure m, we may define the integral of a bounded lower semicontinuous function $g\colon X \to \mathbb{R}_+$ with respect to a continuous valuation μ. Indeed, for every r, the preimage $[g > r] = g^{-1}(]r, +\infty])$ is an open upper set. Thus $\mu\big([g > r]\big)$ is a well defined non-negative real number. Moreover, the function $r \mapsto \mu\big([g > r]\big)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is monotone decreasing and upper semicontinuous. Hence its (Riemann) integral $\int_0^{+\infty} \mu\big([g > r]\big)\, dr$ is a well defined real number. Note that in fact the integral is only extended over the finite interval $[0, \|g\|]$, as $\mu\big([g > r]\big) = 0$ for $r \geq \|g\|$. So we set

$$\int g\, d\mu := \int_0^{+\infty} \mu\big([g > r]\big)\, dr\, .$$

From this one deduces the following properties:

Lemma 23. *The map $(\mu, f) \mapsto \int f d\mu\colon \mathfrak{V}(X) \times \mathrm{LSC}_{+,b}(X) \to \mathbb{R}_+$ is linear and Scott-continuous in each of its two arguments. In detail:*

(i) Let $f \leq g \in \mathrm{LSC}_{+,b}(X)$. Then $\int f\, d\mu \leq \int g\, d\mu$ holds for all $\mu \in \mathfrak{V}(X)$.

(ii) Let $\mu \in \mathfrak{V}(X)$ and assume $(f_i)_{i\in I} \subseteq \mathrm{LSC}_{+,b}(X)$ is directed such that the pointwise supremum f remains bounded. Then $\int f\, d\mu = \sup_{i\in I} \int f_i\, d\mu$ holds.

(iii) Let $r, s \in \mathbb{R}_+$ and $f, g \in \mathrm{LSC}_{+,b}(X)$. Then $\int (rf + sg)\, d\mu = r \int f\, d\mu + s \int g\, d\mu$ holds for all $\mu \in \mathfrak{V}(X)$.

(iv) Let $\mu \preccurlyeq \mu' \in \mathfrak{V}(X)$. Then $\int f\, d\mu \leq \int f\, d\mu'$ holds for all $f \in \mathrm{LSC}_{+,b}(X)$.

(v) Let $f \in \mathrm{LSC}_{+,b}(X)$ and assume $(\mu_i)_{i\in I} \subseteq \mathfrak{V}(X)$ is directed such that the pointwise supremum μ remains bounded. Then $\int f\, d\mu = \sup_{i\in I} \int f\, d\mu_i$.

(vi) Let $r, s \in \mathbb{R}_+$ and $\mu, \mu' \in \mathfrak{V}(X)$. Then $\int f\, d(r\mu + s\mu') = r \int f\, d\mu + s \int f\, d\mu'$ holds for all $f \in \mathrm{LSC}_{+,b}(X)$.

The proof is straightforward except for (iii), for which one employs the approximation of lower semicontinuous functions by simple ones, as stated in Lemma 18. The complete argument can be found in [Tix95] and [Law, Section 3]. We note that the lemma can be shown in more generality, loosening the requirement of boundedness of valuations and functions, see [Kir93]. Also, it is an easy exercise to show that preservation of directed suprema implies monotonicity, so (i) and (iv) are not strictly necessary. However, we wanted to stress that linear Scott-continuous functionals on $\mathrm{LSC}_{+,b}(X)$ are *positive* in the same sense as the elements of $\mathrm{C}^\dagger(X)$ discussed before.

As with measures, we intend to replace valuations by linear functionals on $\mathrm{LSC}_{+,b}(X)$. To begin with, the analogue to the Riesz Representation Theorem is a triviality:

Proposition 24. *Let $(X, \mathcal{G})$ be a topological space. Then for every positive linear Scott-continuous functional on $\mathrm{LSC}_{+,b}(X)$ there is a unique continuous valuation μ such that*

$$\varphi(f) = \int f\, d\mu \quad \textit{for every } f \in \mathrm{LSC}_{+,b}(X)\, .$$

Proof. The characteristic function of an open set belongs to $\mathrm{LSC}_{+,b}(X)$, so the definition of μ is forced on us: $\mu(U) := \varphi(\chi_U)$. It is immediate that we get a bounded continuous valuation this way. In order to see that integration of a lower semicontinuous function g with respect to μ yields φ, we approximate g by a sum of scaled characteristic functions as exhibited in Lemma 18. The statement then follows readily from Scott-continuity of φ. □

We denote the set of all positive linear Scott-continuous functionals on $\mathrm{LSC}_{+,b}(X)$ with $\mathrm{LSC}^{\dagger}_{+,b}(X)$. It is obviously a cone and can be ordered by setting

$$\varphi \preccurlyeq \varphi' \;:\Longleftrightarrow\; \forall g \in \mathrm{LSC}_{+,b}(X).\ \varphi(g) \leq \varphi'(g)\ .$$

We thus get the analogue to Theorem 21, the proof of which is trivial because of the presence of characteristic functions in $\mathrm{LSC}_{+,b}(X)$:

Theorem 25. *For a topological space* $(X, \mathcal{G})$ *the ordered cones* $(\mathfrak{V}(X), \preccurlyeq)$ *and* $(\mathrm{LSC}^{\dagger}_{+,b}(X), \preccurlyeq)$ *are isomorphic.*

3.3 The bijection between measures and valuations

We will now apply the results from the previous two sections to a compact ordered space $(X, \mathcal{O}, \leq)$. Specifically, we will show that the cones $\mathfrak{M}(X)$ of Radon measures and $\mathrm{C}^{\dagger}_{+}(X)$ of positive linear functionals on $\mathrm{C}(X)$, on the one hand, and the cones $\mathfrak{V}(X^{\uparrow})$ of bounded continuous valuations and $\mathrm{LSC}^{\dagger}_{+,b}(X^{\uparrow})$ of linear Scott-continuous functionals on $\mathrm{LSC}_{+,b}(X^{\uparrow})$, on the other hand, are isomorphic. We will also show that the isomorphisms preserve the stochastic orders $\preccurlyeq$ that we defined in each case. This will establish a bijection between Radon measures, which are defined for *all Borel-sets* of $\mathcal{O}$, and valuations, which assign a weight to upper open sets alone. The road map for the proof is given by the following diagram

$$\begin{array}{ccc}
\mathrm{C}^{\dagger}(X) & \longleftrightarrow & \mathrm{LSC}^{\dagger}_{+,b}(X^{\uparrow}) \\
\text{Theorem 21}\ \updownarrow & & \updownarrow\ \text{Theorem 25} \\
\mathfrak{M}(X) & & \mathfrak{V}(X^{\uparrow})
\end{array}$$

Theorem 26. *For a compact ordered space* $(X, \mathcal{O}, \leq)$ *the ordered cones* $(\mathrm{C}^{\dagger}(X), \preccurlyeq)$ *and* $(\mathrm{LSC}^{\dagger}_{+,b}(X^{\uparrow}), \preccurlyeq)$ *are isomorphic.*

Proof. We remind the reader of the function spaces introduced in 2.6 and the inclusions $\mathrm{CM}_{+}(X) \subseteq (\mathrm{CM}_{+} - \mathrm{CM}_{+})(X) \subseteq \mathrm{C}(X)$ and $\mathrm{CM}_{+}(X) \subseteq \mathrm{LSC}_{+,b}(X^{\uparrow})$. The idea of the proof is to show that, on the one hand, *monotone* linear functionals on $\mathrm{CM}_{+}(X)$ are in one-to-one correspondence to *positive* linear functionals on $(\mathrm{CM}_{+} - \mathrm{CM}_{+})(X)$ are in one-to-one correspondence to *positive* linear functionals on $\mathrm{C}(X)$, and on the other hand, *monotone* linear functionals on $\mathrm{CM}_{+}(X)$ are in one-to-one correspondence to *Scott-continuous* linear functionals on $\mathrm{LSC}_{+,b}(X^{\uparrow})$.

Now, working towards the latter equivalence, a Scott-continuous linear functional on $\mathrm{LSC}_{+,b}(X^{\uparrow})$ can obviously be restricted to a monotone linear functional on $\mathrm{CM}_{+}(X)$. Vice versa, we can extend a monotone linear functional φ on $\mathrm{CM}_{+}(X)$ by the formula

$$\overline{\varphi}(f) := \sup\{\varphi(g) \mid g \in \mathrm{CM}_{+}(X) \text{ and } g(x) \leq f(x) \text{ for all } x \in X\}\ ,$$

and the only question is whether the extension is Scott-continuous. To show this, assume that $(f_i)_{i\in I}$ is a directed family of semicontinuous functions, and let $g \in \mathrm{CM}_+(X)$ be such that $g(x) \leq \sup_{i\in I} f_i(x)$ for all $x \in X$. Fix $\epsilon > 0$. For every x we may choose an index $i(x)$ such that $g(x)-\epsilon < f_{i(x)}(x)$. As g is continuous and as $f_{i(x)}$ is lower semicontinuous, there is an open neighbourhood U_x of x such that $g(y) - \epsilon < f_{i(x)}(y)$ for all $y \in U_x$. By compactness, finitely many of the open sets U_x are covering X. Thus, as the f_i form a directed family, we may choose an index i_0 such that $g(x) - \epsilon < f_{i_0}(x)$ for all $x \in X$. Define the function $g_\epsilon \in \mathrm{CM}_+(X)$ by $g_\epsilon(x) = \max\{g(x) - \epsilon, 0\}$ and note that $g_\epsilon \leq f_{i_0}$ holds. From the monotonicity of φ we get that $\varphi(g) - \varphi(g_\epsilon) = \varphi(g - g_\epsilon) \leq \varphi(\epsilon \cdot \mathbf{1}) = \epsilon \cdot \varphi(\mathbf{1})$ and hence $\overline{\varphi}(f_{i_0}) \geq \varphi(g_\epsilon) \geq \varphi(g) - \epsilon \cdot \varphi(\mathbf{1})$. We get $\sup_{i\in I} \overline{\varphi}(f_i) \geq \varphi(g)$ by letting $\epsilon \to 0$.

Restriction and extension are inverses of each other because, on the one hand, $\mathrm{CM}_+(X) \subseteq \mathrm{LSC}_{+,b}(X^\uparrow)$ and, on the other hand, the elements of $\mathrm{LSC}_{+,b}(X^\uparrow)$ are pointwise suprema of elements of $g \in \mathrm{CM}_+(X)$ such that $g(x) \leq f(x)$ for all $x \in X$ by Lemma 17. This latter fact also shows that the stochastic order is translated to the pointwise order of functionals on $\mathrm{CM}_+(X)$.

At the other side, we can likewise restrict a positive linear functional on $\mathrm{C}(X)$ to the cone $\mathrm{CM}_+(X)$ of non-negative order preserving continuous functions. For the extension we first set $\overline{\varphi}(g - g') := \varphi(g) - \varphi(g')$ in order to get a positive linear functional on $(\mathrm{CM}_+ - \mathrm{CM}_+)(X)$. This is well-defined because $g - g' = h - h'$ is equivalent to $g + h' = h + g'$ and φ preserves addition. Positivity and linearity mean that $\overline{\varphi}$ is uniformly continuous with respect to the supremum norm, and therefore we can extend it to a functional on $\mathrm{C}(X)$ by Lemma 19. The extension remains positive and linear.

In this case, too, restriction and extension are inverses of each other because of the density of $(\mathrm{CM}_+ - \mathrm{CM}_+)(X)$ in $\mathrm{C}(X)$. The stochastic order on $\mathrm{C}^\dagger(X)$ is directly defined with reference to $\mathrm{CM}_+(X)$, so the order-theoretic side of the isomorphism needs no further argument. $\square$

Note that *en passant* we have shown that the stochastic preorder on $\mathrm{C}^\dagger(X)$ is antisymmetric.

It remains to interpret what these somewhat involved transformations amount to for measures and valuations. To this end let $U \in \mathfrak{U}$ be an upper open set, and $m \in \mathfrak{M}(X)$ a bounded Radon measure. Because of inner regularity and the order Urysohn property, we find a continuous order preserving function $g\colon X \to [0, 1]$, for which $\varphi(g) = \int g\, dm$ is as close to $m(U)$ as we desire. The value of the corresponding functional on $\mathrm{LSC}_{+,b}(X^\uparrow)$ at χ_U is given as the supremum of the value of φ at these functions and must therefore equal $m(U)$. In other words, the combined translation from $\mathfrak{M}(X)$ to $\mathfrak{V}(X^\uparrow)$ is nothing other than the restriction to open upper sets. Concentrating on its inverse we can thus state:

Theorem 27. *For a compact ordered space $(X, \mathcal{O}, \leq)$, every bounded continuous valuation on $X^\uparrow$ extends uniquely to a Radon measure on X.*

This result is not new; it was first established by Jimmie Lawson, [Law82]. It is also not the most general; see [AM01] and the references given there. However, our proof lends itself particularly well to a discussion of topologies for spaces of valuations and measures, the topic of the next section.

4 Topologies on spaces of measures and valuations

4.1 The vague topology on the space of measures

There are a number of topologies that one could choose for the set of measures. A reasonable minimal requirement is to ask that if a net $(m_i)_{i\in I}$ converges to m then we should also have $\int f\ dm_i \longrightarrow \int f\ dm$ in $\mathbb{R}$. The main free parameter in this condition is the choice of the set of functions from which f may be drawn, and several possibilities are indeed discussed in the literature, e.g. [Top70]. With an eye towards the Riesz Representation Theorem 20, we define:

Definition 28. *Let X be a topological space. The* vague topology $\mathcal{V}$ *on $\mathfrak{M}(X)$ is the weakest topology such that $m \mapsto \int f\ dm \colon \mathfrak{M}(X) \to \mathbb{R}$ is continuous for all $f \in \mathrm{C}(X)$.*

For a compact Hausdorff space we have $\mathfrak{M}(X) \cong \mathrm{C}^\dagger(X)$, and one sees that the vague topology is simply the restriction of what is usually called *weak*-topology* on the dual space $\mathrm{C}^*(X)$ to the cone $\mathrm{C}^\dagger(X)$. We have the following equivalent characterisations in case the underlying space is compact ordered:

Proposition 29. *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space. For a net $(m_i)_{i\in I}$ of bounded Radon measures and a bounded Radon measure m, the following are equivalent:*

(i) $(m_i)_{i\in I}$ *converges to m in the vague topology, that is*

$$\int f\ dm \;=\; \lim_{i\in I} \int f\ dm_i$$

for all $f \in \mathrm{C}(X)$.

(ii) $\int g\ dm_i$ *converges to* $\int g\ dm$ *in $\mathbb{R}$, that is*

$$\int g\ dm \;=\; \lim_{i\in I} \int g\ dm_i$$

for all $g \in \mathrm{CM}_+(X)$.

(iii) $m_i(O)$ *converges to $m(O)$ for all $O \in \mathcal{O}$ in the topology of convergence from below on $\mathbb{R}$, and $m_i(X)$ converges to $m(X)$ in the usual topology on $\mathbb{R}$, that is,*

$$\begin{aligned} m(O) &\leq \liminf_{i\in I} m_i(O) \quad \textit{for all } O \in \mathcal{O}\ , \textit{ and} \\ m(X) &= \lim_{i\in I} m_i(X)\ . \end{aligned}$$

Proof. The direction (i) $\implies$ (ii) being trivial, assume that $\int g\ dm_i$ converges to $\int g\ dm$ for elements of $\mathrm{CM}_+(X)$. Then the integrals will also converge for functions from $(\mathrm{CM}_+ - \mathrm{CM}_+)(X)$ because subtraction is continuous. To extend the statement to all continuous functions f, we employ Lemma 19:

$$\int f\ dm = \lim_{g\to f} \int g\ dm = \lim_{g\to f} \lim_{i\in I} \int g\ dm_i = \lim_{i\in I} \lim_{g\to f} \int g\ dm_i = \lim_{i\in I} \int f\ dm_i\ ,$$

where we have written $g \to f$ to indicate a net of functions from $(\mathrm{CM}_+ - \mathrm{CM}_+)(X)$ converging to f in the supremum norm.

The equivalence with (iii) is part of Topsøe's *Portmanteau Theorem 8.1*, [Top70]. $\square$

Note that $\mathrm{CM}_+(X)$ is a much smaller set of functions than $\mathrm{C}(X)$, and so the fact that it induces the same topology on $\mathfrak{M}(X)$ is remarkable.

Lemma 30. *For a compact ordered space the stochastic order $\preccurlyeq$ on $\mathrm{C}^\dagger(X)$ is closed in the vague topology.*

Proof. Let φ_j and ψ_j be nets of positive linear functionals that converge to φ and ψ, respectively, such that $\varphi_j \preccurlyeq \psi_j$ for every $j \in J$. Then, for every $f \in \mathrm{CM}_+(X)$, we have $\varphi_j(f) \leq \psi_j(f)$ and, as $\varphi_j(f)$ and $\psi_j(f)$ converge to $\varphi(f)$ and $\psi(f)$, respectively, we conclude that $\varphi(f) \leq \psi(f)$, whence $\varphi \preccurlyeq \psi$. $\square$

In [Edw78] it has been shown that, for a compact ordered space, the set of probability measures with the vague topology and the stochastic order is a compact ordered space again. We have a slight generalisation:

Theorem 31. *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space.*

(i) $(\mathfrak{M}(X), \mathcal{V}, \preccurlyeq)$ *is an ordered space.*

(ii) The subsets $\mathfrak{M}_1(X)$ and $\mathfrak{M}_{\leq 1}$ are compact and convex.

Proof. The first claim follows immediately from the preceding lemma. For the second we offer two arguments: Identify (sub)probability measures with positive linear functionals on $\mathrm{C}(X)$, and these in turn with elements in the product $\prod_{f\in \mathrm{C}(X), \|f\|\leq 1}[-1, 1]$. The restriction of the vague topology coincides with the product topology and hence is compact Hausdorff on the full product. Those tuples which correspond to positive linear functionals are characterised by equations and inequalities involving a finite number of coordinates in each instance, hence they define a closed subset.

Alternatively, we can invoke the Banach-Alaoglu Theorem which states that the unit ball in $\mathrm{C}^*(X)$ is compact in the weak* topology. Again, the positive functionals are excised by inequalities and hence form a closed subset. Probability measures are characterised by the single additional requirement $\varphi(\mathbf{1}) = 1$. $\square$

For every $x \in X$, the Dirac functional δ_x, defined by $f \mapsto f(x)$, is a positive linear functional on $\mathrm{C}(X)$. For any completely regular space, $x \mapsto \delta_x$ is a topological embedding of the space X into $\mathrm{C}^*(X)$ endowed with the weak*-topology. In fact, for compact Hausdorff spaces, the functionals δ_x are exactly the extreme points of $\mathrm{C}_1^*(X)$ (see [Cho69, page 108]). We have more:

Proposition 32. *Let X be a compact ordered space. Associating to every element $x \in X$ its Dirac functional δ_x yields a topological and an order embedding of $(X, \mathcal{O}, \leq)$ into $(\mathfrak{M}(X), \mathcal{V}, \preccurlyeq)$.*

Proof. It only remains to show that we have an order embedding. If $x \leq y$, then $\delta_x(f) = f(x) \leq f(y) = \delta_y(f)$ for every $f \in \mathrm{CM}_+(X)$, whence $\delta_x \preccurlyeq \delta_y$. If, on the other hand, $x \not\leq y$, then there is an $f \in \mathrm{CM}_+(X)$ such that $f(x) = 1$ but $f(y) = 0$, that is, $\delta_x(f) = 1 \not\leq 0 = \delta_y(f)$ and, consequently, $\delta_x \not\preccurlyeq \delta_y$. $\square$

4.2 The weak upwards topology on the space of valuations

As with measures, we base our definition of a topology for the set of valuations on integration:

Definition 33. *Let $(X, \mathcal{G})$ be a topological space. The* weak upwards topology $\mathcal{S}$ *on $\mathfrak{V}(X)$ is the weakest topology such that $\mu \mapsto \int g\, d\mu \colon \mathfrak{V}(X) \to \mathbb{R}^{\uparrow}$ is continuous for all $g \in \mathrm{LSC}_{+,b}(X)$.*

Note the use of the topology of convergence from below on $\mathbb{R}$ in this definition.

Proposition 34. *Let $(X, \mathcal{U})$ be a stably compact space. For a net $(\mu_i)_{i\in I}$ of bounded continuous valuations and a bounded continuous valuation μ, the following are equivalent:*

(i) $(\mu_i)_{i\in I}$ converges to μ in the weak upwards topology $\mathcal{S}$, that is

$$\int g\, d\mu \quad \leq \quad \liminf_{i\in I} \int g\, d\mu_i$$

for all $g \in \mathrm{LSC}_{+,b}(X)$.

(ii) $(\int g\, d\mu_i)_{i\in I}$ converges to $\int g\, d\mu$ in $\mathbb{R}^{\uparrow}$, that is

$$\int g\, d\mu \quad \leq \quad \liminf_{i\in I} \int g\, d\mu_i \ ,$$

for all $g \in \mathrm{CM}_+(X_p)$.

(iii) $(\mu_i(U))_{i\in I}$ converges to $\mu(U)$ in $\mathbb{R}^{\uparrow}$, that is

$$\mu(U) \quad \leq \quad \liminf_{i\in I} \mu_i(U) \quad ,$$

for all open sets $U \in \mathcal{U}$.

Proof. Clearly, (i) $\Longrightarrow$ (ii). Further, (i) $\Longrightarrow$ (iii), as the characteristic function χ_U of every open upper set U is lower semicontinuous and $\int \chi_U\, d\mu = \mu(U)$.

(ii) $\Longrightarrow$ (i): By Lemma 17 every $g \in \mathrm{LSC}_{+,b}(X)$ is the supremum of a directed family of monotone increasing continuous functions $f_j \colon X_p \to \mathbb{R}_+$. For the latter we have $\int f_j\, d\mu \leq \liminf_{i\in I} \int f_j\, d\mu_i$ by assumption. As $f_j \leq g$, we have $\liminf_{i\in I} \int f_j\, d\mu_i \leq \liminf_{i\in I} \int g\, d\mu_i$ for all j, whence $\int g\, d\mu = \int \sup_{j\in J} f_j\, d\mu = \sup_{j\in J} \int f_j\, d\mu \leq \sup_{j\in J} \liminf_{i\in I} \int f_j\, d\mu_i \leq \liminf_{i\in I} \int g\, d\mu_i$ as desired. Note that we have used the fact that $f \mapsto \int f\, d\mu$ preserves directed sups as stated in Lemma 23(ii).

(iii) $\Longrightarrow$ (i) is proved in a similar way using the fact that every $g \in \mathrm{LSC}_{+,b}(X)$ is the supremum of an increasing sequence g_n of finite linear combinations of characteristic functions of open sets as stated in Lemma 18. $\square$

As with Proposition 29, note that both $\mathrm{CM}_+(X_p)$ and the characteristic functions associated with the elements of $\mathcal{U}$ are much smaller sets than $\mathrm{LSC}_{+,b}(X)$ in general, yet they define the same topology.

Choosing a constant net $\mu_i = \nu$ in the preceding proposition yields an alternative proof of the order-isomorphism established in Theorem 26:

Corollary 35. *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space. For continuous valuations μ and ν on $\mathcal{U}$, the following are equivalent:*

(i) $\mu \preccurlyeq \nu$, that is, $\mu(U) \leq \nu(U)$ for every open upper set U;

(ii) $\int f d\mu \leq \int f d\nu$ for every $f \in \mathrm{CM}_+(X)$;

(iii) $\int g d\mu \leq \int g d\nu$ for every $g \in \mathrm{LSC}_{+,b}(X^{\uparrow})$.

We observe that the equivalence (i) $\Leftrightarrow$ (iii) remains valid for any ordered topological space.

4.3 Relating the two topologies

In Theorem 26 we established an isomorphism between the cone $\mathfrak{M}(X)$ of bounded Radon measures on a compact ordered space $(X, \mathcal{O}, \leq)$ and the cone $\mathfrak{V}(X^{\uparrow})$ of bounded valuations on the associated stably compact space $X^{\uparrow} = (X, \mathcal{U})$. We can now compare these two cones as topological spaces. Unfortunately, we do not have a general result here, but must restrict ourselves to (sub)probability measures and valuations. On these subsets, the relationship mirrors that between X and $X^{\uparrow}$:

Theorem 36. *Under the isomorphism exhibited in Theorem 26, the upper open sets in* $(\mathfrak{M}_{\leq 1}(X), \mathcal{V}, \preccurlyeq)$ *are precisely the open sets of* $(\mathfrak{V}_{\leq 1}(X), \mathcal{S})$. *The same is true if one restricts further to probability measures and valuations.*

Proof. We know that $(\mathfrak{M}_{\leq 1}(X), \mathcal{V}, \preccurlyeq)$ is a compact ordered space by Theorem 31, and so we can employ Proposition 9. Assume $m_1 \not\preccurlyeq m_2$; then there exists $g \in \mathrm{CM}_+(X)$ with $\int g\, dm_1 > \int g\, dm_2$. Let $K \in \mathbb{R}$ be a number strictly between these two quantities. The sets

$$\begin{aligned} U &:= \{m \in \mathfrak{M}(X) \mid \textstyle\int g\, dm > K\} \quad \text{and} \\ V &:= \{m \in \mathfrak{M}(X) \mid \textstyle\int g\, dm < K\} \end{aligned}$$

are open in the vague topology and disjoint. The first is clearly upwards closed while the second is downwards closed. Furthermore, under the bijection between measures and valuations, U is mapped to the set $\{\mu \in \mathfrak{V}(X) \mid \int g\, d\mu > K\}$ which is weak upwards open by Proposition 34(ii). This shows that upper open sets of $\mathcal{V}$ correspond to weak upwards open sets of valuations. The converse follows directly from Propositions 29(ii) and 34(ii). $\square$

Corollary 37. *Let* $(X, \mathcal{U})$ *be a stably compact space. Then both* $(\mathfrak{V}_{\leq 1}(X), \mathcal{S})$ *and* $(\mathfrak{V}_1(X), \mathcal{S})$ *are again stably compact.*

This result can also be shown directly, without employing any functional analytic methods, as we will now explain. We show more generally that, for a stably compact space X, the set $\overline{\mathfrak{V}}(X)$ of all continuous valuations is again stably compact for the weak upwards topology. We start with the stably compact space $P = \prod_{O \in \mathcal{U}} \overline{\mathbb{R}}_+^{\uparrow}$, where each copy of $\overline{\mathbb{R}}_+$ is equipped with the topology of continuity from below. The corresponding patch topology is just the product topology of the usual compact Hausdorff topology. The set $\mathfrak{m}\mathfrak{V}(X)$ of all (not necessarily continuous) valuations $\mu\colon \mathcal{U} \to \overline{\mathbb{R}}_+$ is patch closed in P, as one easily verifies. By invoking Proposition 15 we have thus shown that the set $\mathfrak{m}\mathfrak{V}(X)$ of valuations on a stably compact space X is stably compact when equipped with the restriction of the product topology.

In order to restrict further to *continuous* valuations, we remember that $(\mathcal{U}, \subseteq)$ is a continuous lattice. We use the following standard technique from domain theory in order to be able to apply Proposition 16:

Proposition 38. *Let* $(X, \mathcal{U})$ *be a stably compact space and* $\mu\colon \mathcal{U} \to \overline{\mathbb{R}}_+$ *be a valuation. The following defines the largest continuous valuation below* μ *in the pointwise order:*

$$\Phi(\mu)(O) := \sup\{\mu(V) \mid V \ll O\}$$

where $V \ll O$ *means that there is a compact saturated set* K *such that* $V \subseteq K \subseteq O$. *Furthermore, the operation* $\Phi\colon \mathfrak{m}\mathfrak{V}(X) \to \mathfrak{m}\mathfrak{V}(X)$ *is idempotent and continuous with respect to the product topology, and maps (sub-)probability valuations to (sub-)probability valuations.*

Proof. It is clear that $\Phi(\mu)(\emptyset) = 0$ holds, and that $\Phi(\mu)$ is monotone. For the modular law, we exploit stable compactness which gives us that $O \cap O'$ is approximated by sets of the form $V \cap V'$ where $V \ll O$ and $V' \ll O'$. The continuity of $\Phi(\mu)$ follows from its definition.

A continuous valuation is kept fixed by Φ because every open set equals the directed union of those open sets way-below it.

In order to see that the operation of making a valuation continuous is itself continuous with respect to the product topology on $\mathfrak{m}\mathfrak{V}(X)$, observe that $\Phi(\mu)(O)$ is greater than a real number r, if and only if $\mu(V) > r$ for some $V \subseteq K \subseteq O$. Hence the preimage of the subbasic open set $\{\mu \in \mathfrak{m}\mathfrak{V}X \mid \mu(O) > r\}$ equals $\bigcup_{V \subseteq K \subseteq O} \{\mu \in \mathfrak{m}\mathfrak{V}(X) \mid \mu(V) > r\}$.

The last statement follows immediately from the fact that the whole space X is compact and open at the same time. $\square$

We thus have by Proposition 16 that the restriction of the product topology to those tuples which correspond to continuous valuations is stably compact. Finally, by Proposition 34(iii) the product topology restricted to the set of (sub-)probability valuations is the same as the weak upwards topology.

Theorem 39. *The set $\mathfrak{V}_{\leq 1}(X)$ of continuous probability valuations on a stably compact space X is stably compact when equipped with the weak upwards topology $\mathcal{S}$. The same holds for $\mathfrak{V}_1(X)$.*

5 Open problems

As we remarked briefly before stating Theorem 36, we do not have a general result relating the vague topology on $\mathfrak{M}(X)$ to the weak upwards topology on $\mathfrak{V}(X^{\uparrow})$, even for very well-behaved topological spaces X. The criterion of success would be if one could derive Theorem 36 as a simple corollary.

As we explained in Section 2.4, domains are characterised by the property that the topology can be derived from the order relation alone. It was shown in [Jon90] that for a domain the set of subprobability valuations together with the stochastic order is again a domain, and it was shown in [Tix95] that the weak upwards topology is the Scott topology in this situation. Now even if the specialisation order of a given stably compact space $(X, \mathcal{U})$ is too sparse to determine the topology, the stochastic order on $\mathfrak{V}_{\leq 1}(X)$ is always quite rich, and there is a possibility that it might suffice to define the weak upwards topology order-theoretically. We leave this question, too, as an open problem.

Finally, we have restricted ourselves to *bounded* measures and valuations throughout. There is a certain price to pay for this because as a result the sets $(\mathfrak{M}(X), \preccurlyeq)$ and $(\mathfrak{V}(X), \preccurlyeq)$ are not directed complete. While we know that some of our lemmas hold for the more general setting where ∞ is allowed as a value, for example 17 and 18, we do not know how to prove the main results in the general setting.

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