06-21254 *Mathematical Techniques for Computer Science*
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Solutions to first in-class test

Exercise 4.1

(a)

$$\left(\begin{array}{rrr|r} 1 & 2 & 1 & -1 \\ 2 & -1 & 1 & 1 \\ -2 & 1 & -2 & 2 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 2 & 1 & -1 \\ 0 & -5 & -1 & 3 \\ 0 & 5 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

The unique solution is $x_1 = 2$, $x_2 = 0$, and $x_3 = -3$.

(b)

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 3 \\ 1 & 0 & -1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 3 \\ 0 & -1 & -2 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution is $x_1 = 3 + x_3, x_2 = -3 - 2x_3$ and x_3 can be chosen freely.

Exercise 4.2

(a) $X = P_1 + s \cdot \overrightarrow{P_1P_2} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + s \cdot \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}$

(b) $X = Q_1 + s \cdot \overrightarrow{Q_1Q_2} + t \cdot \overrightarrow{Q_1Q_3} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + s \cdot \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(c) Normal: $\vec{n} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ as computed by the formula which gives the equation $2x_1 - x_2 - x_3 = 0$ where the right-hand side 0 is computed as $2 \cdot 2 - 2 \cdot 1 - 2 \cdot 1$ by substituting the coordinates of Q_1 into the left-hand side.

(d) $\langle P_1, \vec{n} \rangle = 3$ and the distance to E is $\frac{3}{\sqrt{6}}$. For P_2 we get $\langle P_2, \vec{n} \rangle = -3$, and again, the distance to E is therefore $-\frac{3}{\sqrt{6}}$. They are on opposite sides of E because the signs of the distances are different (one positive, one negative).

(e) Substituting the parametric presentation of the line into $2x_1 - x_2 - x_3 = 0$ yields: $3 - 6s = 0$ from which we get $s = \frac{1}{2}$, so the point H of intersection is $\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

(f) Reflecting $A = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ yields $A' = A + 2 \times \frac{0 - \langle A, \vec{n} \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \frac{2 \times (-3)}{6} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$. For the other point we take the intersection point $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ which does not need to be reflected. This yields the parametric form of the reflected line $X = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ for the reflected line.

(g) The angle α between line and normal to the plane is $\cos \alpha = \frac{\langle \vec{v}, \vec{n} \rangle}{|\vec{v}| \cdot |\vec{n}|} = \frac{-6}{\sqrt{24} \cdot \sqrt{6}} = \frac{-6}{\sqrt{144}} = \frac{-6}{12} = -\frac{1}{2}$ so $\alpha = 120^0$. Now, there are always two angles between two lines and one of them is always less than or equal to 90^0 and the other 180^0 minus that one. In other words, the other angle is $180^0 - 120^0 = 60^0$. The angle between the *plane* itself and the line L is now $90^0 - 60^0 = 30^0$.