The University of Birmingham
School of Computer Science
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Solutions to first in-class test

Question 1

$$\left(\begin{array}{rrr|r} 2 & -3 & 1 & -1 \\ -3 & 2 & 2 & 0 \\ 1 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 2 & -3 & 1 & -1 \\ -3 & 2 & 2 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -1 & 5 & 3 \end{array}\right) \to$$

$$\left(\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 6 & 6 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The unique solution is $x_1 = 2$, $x_2 = 2$, and $x_3 = 1$.

Question 2

The pivots are in the second and third column. The other two variables can be chosen freely. We can describe this as follows:

$$\begin{array}{lcl} x_4 & : & \text{choose freely} \\ x_3 & = & 1+3x_4 \\ x_2 & = & \frac{1}{2}\times(x_3-x_4) = \frac{1}{2}\times(1+3x_4-x_4) = \frac{1}{2}+x_4 \\ x_1 & : & \text{choose freely} \end{array}$$

Question 3

The three sides of the triangle are $\overrightarrow{P_1P_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\overrightarrow{P_2P_3} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, and $\overrightarrow{P_3P_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, each of which has length $\sqrt{1^2+(-1)^2} = \sqrt{2}$.

Question 4

(a) $X = P_1 + s\cdot\overrightarrow{P_1P_2} + t\cdot\overrightarrow{P_1P_3} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + s\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

(b) We get the system of linear equations $\begin{array}{rrrcrr} 1 & -s & -t & = & q & +r \\ -1 & +s & & = & -2q & -2r \\ 1 & & +t & = & 2q & \end{array}$ which we solve by Gaussian elimination:

$$\left(\begin{array}{rrrr|r} -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & -2 & 0 & -1 \end{array}\right) \to \left(\begin{array}{rrrr|r} -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & -1 \end{array}\right) \to \left(\begin{array}{rrrr|r} -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{array}\right)$$

From this we see that r can be chosen freely and q computes as $1+r$. Substituting this back into the equation for E_2 gives $X = (1+r)\cdot\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + r\cdot\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$.

Question 5

(a) Using the given formula to compute the normal to the two direction vectors yields $\vec{n} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ which can be simplified to $\vec{n} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ because the length of the normal vector doesn't matter. Setting up the normal form as $\langle\vec{n},\vec{x}\rangle = \langle\vec{n},P\rangle$

with $P = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ gives us $2x_1 + x_2 = 0$ as the normal form for E_2.

(b) We compute the distance for each of the three points (note that $d = 0$ in our case):

$$-\frac{\langle P_1, \vec{n}\rangle}{|\vec{n}|} = -\frac{2-1}{\sqrt{2^2+1^2}} = \frac{-1}{\sqrt{5}} \qquad -\frac{\langle P_2, \vec{n}\rangle}{|\vec{n}|} = \frac{0}{\sqrt{5}} = 0 \qquad -\frac{\langle P_3, \vec{n}\rangle}{|\vec{n}|} = \frac{1}{\sqrt{5}}$$

We see that P_1 and P_3 are on opposite sides of E_2 while P_2 is actually on E_2.

Question 6

(a) We substitute the line equation into the normal form and obtain $-2s + 2 + 3s = 0$ or $s = -2$. This gives us the intersection point $A = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix}$

(b) We already have the intersection point A which does not need to be reflected. For a second point on L we can choose $B = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ for which the reflected point computes as

$$B' = B + 2 \times \frac{d - \langle B, \vec{n}\rangle}{\langle \vec{n}, \vec{n}\rangle} \cdot \vec{n} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + 2 \times \frac{0-2}{5} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} -8 \\ 6 \\ 5 \end{pmatrix}$$

The equation for the line through A and B' is given as

$$X = A + s \cdot \overrightarrow{AB'} = \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix} + s \times \frac{1}{5} \cdot \begin{pmatrix} -18 \\ 26 \\ 10 \end{pmatrix}$$

which can be simplified to (since the length of the direction vector does not matter):

$$X = \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix} + s \cdot \begin{pmatrix} -9 \\ 13 \\ 5 \end{pmatrix}$$

Question 7

(a) The point P' is nearest to P if the line connecting P and P' is orthogonal to the direction of the given line. Orthogonality can easily be tested with the inner product (which must yield 0 for orthogonal vectors).

(b) For P' on the given line L we have the coordinates

$$P' = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

Therefore the vector from P to P' has coordinates

$$\overrightarrow{PP'} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

The inner product with $\vec{v} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ gives the equation $\left\langle \vec{v}, \overrightarrow{PP'} \right\rangle = 0$, or substituted with actual values $\left\langle \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -s \\ 2+3s \\ s \end{pmatrix} \right\rangle = 0$. We get $s + 6 + 9s + s = 0$ and hence $s = -\frac{6}{11}$, so $P' = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \frac{6}{11} \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{11} \cdot \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}$.