06-21254 *Mathematical Techniques for Computer Science*
Autumn Semester 2010
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School of Computer Science
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Solutions and mark sheet for first in-class test

Question 1

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 3 & 0 & -2 & 3 \\ 1 & 1 & 0 & -1 \end{array}\right)$$ 1 point

Exchange the first and third line:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 3 & 0 & -2 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right)$$ 1 point

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -3 & -2 & 6 \\ 0 & 1 & 1 & 1 \end{array}\right)$$ 1 point

Exchange second and third line:

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -2 & 6 \end{array}\right)$$ 1 point

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 9 \end{array}\right)$$ 1 point

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 9 \end{array}\right)$$ 1 point

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 9 \end{array}\right)$$ 1 point

Solutions: $x_1 = 7, x_2 = -8, x_3 = 9$.

Question 2

(a)

$$x_1 : \text{freely chosen}$$ 1 point

$$x_2 : \text{freely chosen}$$ 1 point

$$x_3 = 0$$ 1 point

(b) Geometrically, the solutions form a plane in 3D. 1 point

Question 3

(a) $\vec{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ 1 point

(b) Using the formula for the distance of a point from a plane one obtains:

$$\frac{d - \langle \vec{n}, P_1 \rangle}{|\vec{n}|} = \frac{1-(3-4+3)}{\sqrt{6}} = \frac{-1}{\sqrt{6}}$$ 1 point

$$\frac{d - \langle \vec{n}, P_2 \rangle}{|\vec{n}|} = \frac{1-(2-2)}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$ 1 point

Because the magnitude is the same, they are equally far away from the plane; because the signs are different (one positive, one negative), they are on opposite sides. 1 point

(c) $X = P_1 + s \cdot \overrightarrow{P_1P_2} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + s \cdot \begin{pmatrix} -1 \\ -2 \\ -5 \end{pmatrix}$ 2 points

(d) We substitute the coordinates of the line into the equation for the plane: 1 point

$$(3-s) - 2(2-2s) + (3-5s) = 1$$

and solve for s: 1 point

$$\begin{aligned} 3-4+3+s(-1+4-5) &= 1 \\ -2s &= -1 \\ s &= \tfrac{1}{2} \end{aligned}$$

We substitute this value of s into the parametric representation and obtain the intersection point: 1 point

$$P_m = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} -1 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 1 \\ 0.5 \end{pmatrix}$$

(Note: Since we know that the two points are equally far away from the plane, we could immediately have concluded that intersection point must be the midpoint between P_1 and P_2. This can easily be computed by taking the average in each coordinate.)

(e) We need to mirror two points of the line but since we have the intersection point, one is enough. (In other words, the intersection point lies on the plane and doesn't change by mirroring.) 1 point

We mirror P_1:

$$P_1' = P_1 + 2\frac{d - \langle \vec{n}, P \rangle_1}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + 2\frac{-1}{6} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{8}{3} \\ \frac{8}{3} \end{pmatrix}$$ 1 point

By connecting P_1' and P_m we get the parametric representation of the mirrored line:

$$P_1' + s \cdot \overrightarrow{P_1'P_m} = \begin{pmatrix} \frac{8}{3} \\ \frac{8}{3} \\ \frac{8}{3} \end{pmatrix} + s \cdot \begin{pmatrix} -\frac{1}{6} \\ -\frac{5}{3} \\ -\frac{13}{6} \end{pmatrix}$$ 1 point

which can be simplified by moving along the line (s=-2) and using a longer direction vector:

$$\begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 10 \\ 13 \end{pmatrix}$$

If you choose to mirror P_2 instead of making use of the intersection point, then you get

$$P_2' = \begin{pmatrix} \frac{7}{3} \\ -\frac{2}{3} \\ -\frac{5}{3} \end{pmatrix}$$

Total points: 23