

Solutions and mark sheet for the second in-class test

Question 1

- (a) $A \cup B = \{-3, -2, -1, 1, 2, 3\}$ 2 points
- (b) $A \cap B = \emptyset$ 2 points
- (c) $A \setminus B = A = \{1, 2, 3\}$ 2 points
- (d) $A \times B = \{(1, -1), (1, -2), (1, -3), (2, -1), (2, -2), (2, -3), (3, -1), (3, -2), (3, -3)\}$ 2 points

Question 2

- (a) True, because $2 + (-1) = 1$ which is between -3 and 3 . 1 point
- (b) True, because $2^2 + (-1)^2 = 4 + 1 = 5 \leq 5$. 1 point
- (c) False, 1 point
because $(0, 3) \in A$ but $(0, 3) \notin B$. 1 point
- (d) True, 1 point
because B is the set of points with integer coordinates whose distance from the origin is less than or equal to $\sqrt{5}$, and largest coordinates one could therefore have are $(1, 2)$ or $(2, 1)$, both of which belong to A . 1 point
- (e) False, 1 point
because $(2, -1)$ belongs to both A and B . 1 point

Question 3

- (a) The lists of finite length can be enumerated as follows: First the empty list, then the two lists of length 1, then the four lists of length 2, then the eight lists of length 3, etc., etc. 3 points
(Note that the lists that only contain 0 or 1 is really the same as the set of words over the alphabet $\{0, 1\}$, which we considered in class.)
- (b) The proof is by contradiction, and very similar to (but actually simpler than) the proof we did for the real numbers in class. Assuming an enumeration of infinite lists we construct a “diagonal list” d whose n -th entry is $1 - a_n^n$ where a_n^n is the n -th entry of the n -th list in the enumeration. By construction, the diagonal list d is different from every list in the enumeration, because we have made sure that the n -th element of d is the opposite of the n -th element of the list in n -th position in the enumeration. 3 points

Question 4

- (a) The relation is not reflexive because (c, c) and (d, d) are missing from R . 1 point
It is symmetric because with every $(x, y) \in R$, the pair (y, x) is also in R . 1 point
It is also transitive, as one can check by going through all the possible combinations of the form $(x, y), (y, z)$ (there are only four). 1 point
- (b) We only need to add the missing reflexive pairs: $R' = R \cup \{(c, c), (d, d)\}$. 2 points

(c) There are three equivalence classes:

$$\begin{aligned} [a]_{R'} &= [b]_{R'} = \{a, b\} \\ [c]_{R'} &= \{c\} \\ [d]_{R'} &= \{d\} \end{aligned}$$

3 points

Question 5

(a) Assuming that there is no limit to the size of a `String` object in Java, this is a perfectly good function: for every input it gives one and only one result.

2 points

It is injective, as two different strings remain different if both get extended by the string "aa".

1 point

It is not surjective because the output can never be a string of less than two characters.

1 point

(b) It is difficult to say how a concrete Java implementation would behave here. If we assume that very long `String` objects are possible then their length could be longer than what fits into an `int`, in which case `length()` might raise an exception. If we assume that no exception will be raised (or that `String` objects can never be longer than $2^{31} - 1$, then we get a function in the mathematical sense.

2 points

The function is definitely not injective because there are different strings of the same length.

1 point

The function is also not surjective because strings of negative length are not possible.

1 point

(c) This is not a function because for the same input it gives different outputs, depending on the time of day, so it is not single-valued.

2 points

It does give an output for every input, though.

1 point

Injectivity and surjectivity make no sense if we don't have a function.

1 point

Question 6

(a) The relation is reflexive, because for A and $A' = A$ we can choose $B = E$, the 2×2 identity matrix for which we have $E^{-1} = E$, so $A = EAE^{-1}$.

2 points

For symmetry, assume $A \approx A'$ with $A' = BAB^{-1}$. We multiply this equation with B^{-1} on the left and with $(B^{-1})^{-1}$ on the right and get $B^{-1}A'(B^{-1})^{-1} = B^{-1}BAB^{-1}(B^{-1})^{-1} = EAE = A$, which shows $A' \approx A$. (The only doubt one might have about this proof is whether B^{-1} is invertible, but this is the case and the inverse is in fact just B .)

2 points

For transitivity assume $A \approx A'$ and $A' \approx A''$, that is, there are invertible matrices B and C such that $A' = BAB^{-1}$ and $A'' = CA'C^{-1}$. By simple substitution one gets $A'' = CBAB^{-1}C^{-1}$ and all that remains is to show that $B^{-1}C^{-1}$ is the inverse to CB , but this is easy: $CBB^{-1}C^{-1} = CEC^{-1} = E$.

2 points

(b) Consider the identity matrix E . If A' is similar to E then it must be the case that $A' = BEB^{-1}$ but the latter evaluates to $BB^{-1} = E$, so the identity matrix is in an equivalence class all by itself; every other matrix is dissimilar to it. (Studying this some more, one can see that most equivalence classes have infinitely many elements, and that there are infinitely many classes.)

2 points

Total points: 50