06-21254 *Mathematical Techniques for Computer Science* — The University of Birmingham
Autumn Semester 2011 — School of Computer Science

Solutions to Exercise Sheet 1

Exercise 1.1

(a)

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 1 & -1 & 4 & 0 \\ 2 & 1 & 5 & 3 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & -3 & -1 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & -2 & 2 \end{array}\right) \to$$

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

...and we may read off the solution $x_1 = 4$, $x_2 = 0$, and $x_3 = -1$.

(b)

$$\left(\begin{array}{rrr|r} 2 & 2 & 1 & -1 \\ -1 & -1 & 2 & 3 \\ 1 & 1 & 3 & 3 \end{array}\right) \to \left(\begin{array}{rrr|r} 2 & 2 & 1 & -1 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 5 & 7 \end{array}\right) \to \left(\begin{array}{rrr|r} 2 & 2 & 1 & -1 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

There is no solution because the last line indicates a contradiction: $0 = 2$.

(c)

$$\left(\begin{array}{rrr|r} -4 & -1 & -1 & -3 \\ 2 & -2 & -1 & 4 \\ 2 & 3 & 2 & -1 \end{array}\right) \to \left(\begin{array}{rrr|r} -4 & -1 & -1 & -3 \\ 0 & -5 & -3 & 5 \\ 0 & 5 & 3 & -5 \end{array}\right) \to \left(\begin{array}{rrr|r} -4 & -1 & -1 & -3 \\ 0 & -5 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{solution:}\quad \begin{array}{rcl} x_3 & : & \text{chosen freely} \\ x_2 & = & (5+3x_3)/(-5) = -1 - \frac{3}{5}x_3 \\ x_1 & = & (-3+x_3+x_2)/(-4) = (-3+x_3-1-\frac{3}{5}x_3)/(-4) = 1 - \frac{1}{10}x_3 \end{array}$$

Exercise 1.2

It helps to "envelop" the non-zero values with a "staircase":

$$\text{(i)}\ \left(\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{(ii)}\ \left(\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{(iii)}\ \left(\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

The staircase drops never more than by one step, so in all three cases we do have the required echelon form. In (i) and (iii) we get a contradictory equation below the the last step of the staircase ($0 = 3$ in both cases), so these systems are contradictory and have no solution. In case (ii) we see that the staircase doesn't drop in the second column, so x_2 can be chosen freely; we get the following description of the solution set in this case:

$$\begin{array}{rcl} x_3 & = & 3 \\ x_2 & = & \text{chosen freely} \\ x_1 & = & 2-3+x_2 = x_2 - 1 \end{array}$$

Exercise 1.3

The "Special Base Case 1" arises when a contradiction is detected. This can not happen for a homogeneous system as $x_1 = x_2 = \ldots = x_n = 0$ is always a solution. You can also see this by considering how the Gaussian elimination algorithm transforms the equations: the right hand sides always stay equal to zero, so a contradiction can not arise.