Solutions to Exercise Sheet 2

Exercise 2.1

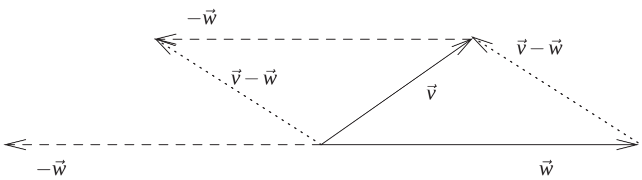
We compute the length of each side:

$$|\overrightarrow{P_1P_2}| = \sqrt{(0-1)^2+(1-0)^2+(1-2)^2} = \sqrt{1+1+1} = \sqrt{3}$$
$$|\overrightarrow{P_2P_3}| = \sqrt{(-1-0)^2+(-3-1)^2+(2-1)^2} = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$$
$$|\overrightarrow{P_3P_1}| = \sqrt{(1-(-1))^2+(0-(-3))^2+(2-2)^2} = \sqrt{4+9+0} = \sqrt{13}$$

and find that they are all different.

Exercise 2.2

(a)



(b) (i)

$$\begin{aligned}
\vec{v}-\vec{v} &= \vec{v}+(-1)\cdot\vec{v} && \text{by definition}\\
&= (1)\cdot\vec{v}+(-1)\cdot\vec{v} && \text{1st rule in Box 26}\\
&= (1+(-1))\cdot\vec{v} && \text{4th rule in Box 26}\\
&= (0)\cdot\vec{v} && \text{ordinary arithmetic}\\
&= \vec{0} && \text{2nd rule in Box 26}
\end{aligned}$$

(ii)

$$\begin{aligned}
s\cdot(\vec{v}-\vec{w}) &= s\cdot(\vec{v}+(-1)\cdot\vec{w}) && \text{by definition}\\
&= s\cdot\vec{v}+s\cdot((-1)\cdot\vec{w}) && \text{5th rule in Box 26}\\
&= s\cdot\vec{v}+(s\times(-1))\cdot\vec{w} && \text{last rule in Box 26}\\
&= s\cdot\vec{v}+((-1)\times s)\cdot\vec{w} && \text{ordinary arithmetic}\\
&= s\cdot\vec{v}+(-1)\cdot(s\cdot\vec{w}) && \text{last rule in Box 26}\\
&= s\cdot\vec{v}-(s\cdot\vec{w}) && \text{by definition}
\end{aligned}$$

Exercise 2.3

(a) We get the system of linear equations (one for each of the three coordinates):

$$\begin{aligned}
s &= 5-2t\\
2s &= -4+3t\\
-1 &= -3+t
\end{aligned}$$

The last equation says that t must equal 2, which gives intersection point $\begin{pmatrix}1\\2\\-1\end{pmatrix}$. To see that this point is indeed on the first line, we solve the system of equation for s. The first equation says $s=5=4=1$ and the second equation $2s=-4+6=2$, that is, also $s=1$. So indeed, $s=1$ and $t=2$ solves all three equations simultaneously.

(b) We go about this one exactly as we did in the previous question:

$$\begin{aligned}
1+s &= 2-2t\\
2s &= -4+t\\
-1+s &= -3+t
\end{aligned}$$

We re-arrange and get

$$\begin{array}{rcl} s+2t & = & 1 \\ 2s-t & = & -4 \\ s-t & = & -2 \end{array}$$

which we attack with Gaussian elimination:

$$\left(\begin{array}{rr|r} 1 & 2 & 1 \\ 2 & -1 & -4 \\ 1 & -1 & -2 \end{array}\right) \rightarrow \left(\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & -5 & -6 \\ 0 & -3 & -3 \end{array}\right) \rightarrow \left(\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & -5 & -6 \\ 0 & -15 & -15 \end{array}\right) \rightarrow \left(\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & -5 & -6 \\ 0 & 0 & 3 \end{array}\right)$$

We obtain a contradictory equation ($0 = 3$) which shows that the system of equations has no solution. Geometrically, it means that the two lines do not intersect.

Exercise 2.4

(a) A parametric representation can be computed as

$$X = P_3 + s \cdot \overrightarrow{P_3P_1} + t \cdot \overrightarrow{P_3P_2} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + s \cdot \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$$

(but that's not the only possibility).

(b) Equating the expression for the plane with that for the line one obtains the vector equation

$$\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + s \cdot \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + r \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

which leads to the three linear equations

$$\begin{array}{rcrcrcrcr} & - & 2s & - & 2t & = & & & 2r \\ 2 & - & 2s & - & 3t & = & 1 & + & r \\ 2 & + & 2s & - & t & = & 3 & + & r \end{array}$$

which we rewrite into standard form

$$\begin{array}{rcrcrcr} -2s & - & 2t & - & 2r & = & 0 \\ -2s & - & 3t & - & r & = & -1 \\ 2s & - & t & - & r & = & 1 \end{array}$$

and apply Gaussian elimination

$$\left(\begin{array}{rrr|r} -2 & -2 & -2 & 0 \\ -2 & -3 & -1 & -1 \\ 2 & -1 & -1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} -2 & -2 & -2 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & -3 & -3 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} -2 & -2 & -2 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -6 & 4 \end{array}\right)$$

From the last line we get $r = -\frac{2}{3}$ (the answers for s and t are not needed) which we enter into the equation for the line to get the intersection point M:

$$M = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - \frac{2}{3} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} \\ \frac{1}{3} \\ \frac{7}{3} \end{pmatrix}$$