06-21254 *Mathematical Techniques for Computer Science* — The University of Birmingham
Autumn Semester 2011 — School of Computer Science

Solutions to Exercise Sheet 4

Exercise 4.1

$$(A+B)^3 = (A+B)((A+B)(A+B)) = (A+B)(AA+AB+BA+BB) = AAA+AAB+ABA+ABB+BAA+BAB+BBA+BBB$$
$$= A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3$$

The thing to note is that no further simplification is possible because matrix multiplication is not commutative.

Exercise 4.2

$$AB = \begin{pmatrix} -12 & 1 \\ -4 & 1 \end{pmatrix} \qquad BA = \begin{pmatrix} -1 & 4 & -5 \\ 1 & -2 & 3 \\ -3 & 5 & -8 \end{pmatrix}$$

Exercise 4.3

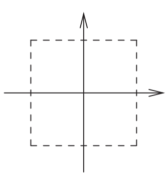
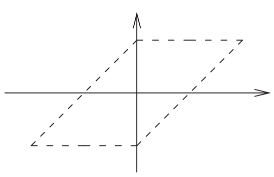
(a) Following the formula for matrix multiplication we get

$$AA = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad AAA = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad AAAA = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \quad AAAAA = \begin{pmatrix} 3 & 5 \\ 5 & 8 \end{pmatrix}$$

(b) All entries of A^n are Fibonacci numbers. If we say $f_0 = 0$ and $f_1 = 1$ for the first two Fibonacci numbers, then A^n has the form $\begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}$

Exercise 4.4

$$AP_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad AP_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad AP_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad AP_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$A(AP_1) = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \qquad A(AP_2) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad A(AP_3) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad A(AP_4) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$A(A(AP_1)) = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \qquad A(A(AP_2)) = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \qquad A(A(AP_3)) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad A(A(AP_4)) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Geometrically, multiplying with the matrix A amounts to a “shearing movement” parallel to the x-axis which pushes points above the x-axis to the right and those below to the left.