Solutions to Exercise Sheet 5

Exercise 5.1

We do the extended Gaussian elimination procedure:

$$\left(\begin{array}{rrr|rrr} 1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -1 & 1 & 0 \\ 0 & 4 & 1 & 2 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -1 & 1 & 0 \\ 0 & 12 & 3 & 6 & 0 & 3 \end{array}\right) \Longrightarrow$$

$$\left(\begin{array}{rrr|rrr} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 3 \end{array}\right) \Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & -3 & -3 & -3 \\ 0 & 0 & 1 & -2 & -4 & -3 \end{array}\right) \Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -2 & -3 & -3 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & -4 & -3 \end{array}\right)$$

So

$$A^{-1} = \begin{pmatrix} -2 & -3 & -3 \\ 1 & 1 & 1 \\ -2 & -4 & -3 \end{pmatrix}$$

Checking the answer:

$$\begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & -1 \\ -2 & -2 & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 & -3 \\ 1 & 1 & 1 \\ -2 & -4 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 & -3 & -3 \\ 1 & 1 & 1 \\ -2 & -4 & -3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & -1 \\ -2 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 5.2

The question makes sense because a 3×3 matrix multiplied with a 3×4 matrix results in a 3×4 matrix.

We multiply the equation $AX = B$ with A^{-1} on the left and obtain $A^{-1}AX = A^{-1}B$ which simplifies to $EX = A^{-1}B$ and $X = A^{-1}B$. So we have to compute the product of A^{-1} with B. This gives

$$X = A^{-1}B = \begin{pmatrix} -2 & -3 & -3 \\ 1 & 1 & 1 \\ -2 & -4 & -3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -1 & 4 \\ -2 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -5 & -2 & -21 \\ -1 & 1 & 1 & 7 \\ 4 & -8 & -1 & -25 \end{pmatrix}$$

Exercise 5.3

Multiplying the second row of A with the second column of B should give the entry e_{22} of the identity matrix, which is 1. We get the equation $\frac{1}{7}(4 \times 1 + ? \times 1 + 2 \times 1) = 1$ from which we conclude $? = 1$. Check:

$$\frac{1}{7} \cdot \begin{pmatrix} 3 & -1 & -2 \\ 4 & 1 & 2 \\ -2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -2 & 1 & -1 \end{pmatrix} = \frac{1}{7} \cdot \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 5.4

The fourth LED adds another column to the table from the beginning of this handout:

	LED red	LED green	LED blue	New LED
eye red	4	2	1	4
eye green	3	4	2	5
eye blue	1	2	4	1

This leads to a 3×4-matrix and so we cannot hope to find an inverse. Instead we compute the required illumination column $p=\begin{pmatrix}p_1\\p_2\\p_3\\p_4\end{pmatrix}$ in the matrix equation $Ap=c$ as described in Item 49, that is, by Gaussian elimination applied to the system of equations

$$\begin{aligned}4p_1+2p_2+1p_3+4p_4 &= 4\\ 3p_1+4p_2+2p_3+5p_4 &= 4\\ 1p_1+2p_2+4p_3+p_4 &= 1\end{aligned}$$

$$\left(\begin{array}{cccc|c}4&2&1&4&4\\3&4&2&5&4\\1&2&4&1&1\end{array}\right)\longrightarrow\left(\begin{array}{cccc|c}1&2&4&1&1\\3&4&2&5&4\\4&2&1&4&4\end{array}\right)\longrightarrow\left(\begin{array}{cccc|c}1&2&4&1&1\\0&-2&-10&2&1\\0&-6&-15&0&0\end{array}\right)\longrightarrow$$

$$\left(\begin{array}{cccc|c}1&0&-6&3&2\\0&-2&-10&2&1\\0&0&15&-6&-3\end{array}\right)\longrightarrow\left(\begin{array}{cccc|c}5&0&-30&15&10\\0&6&30&-6&-3\\0&0&15&-6&-3\end{array}\right)\longrightarrow\left(\begin{array}{cccc|c}5&0&0&3&4\\0&6&0&6&3\\0&0&15&-6&-3\end{array}\right)\longrightarrow$$

$$\left(\begin{array}{cccc|c}1&0&0&3/5&4/5\\0&1&0&1&1/2\\0&0&1&-2/5&-1/5\end{array}\right)$$

From this we see that p_4 can be chosen freely and the other three variables compute to

$$\begin{aligned}p_1 &= 4/5-3p_4/5\\ p_2 &= 1/2-p_4\\ p_3 &= -1/5+2p_4/5\end{aligned}$$

We know that none of the variables is allowed to be negative, so we have the inequalities:

$$\begin{aligned}0 &\le 4/5-3p_4/5\\ 0 &\le 1/2-p_4\\ 0 &\le -1/5+2p_4/5\\ 0 &\le p_4\end{aligned}$$

The second inequality can be rearranged as $p_4\le 1/2$, and the third as $1/5\le 2p_4/5$ or $1/2\le p_4$. This forces $p_4=1/2$. Luckily, this value makes the first and fourth inequality true as well and we get the illumination vector $p=\begin{pmatrix}1/2\\0\\0\\1/2\end{pmatrix}$.