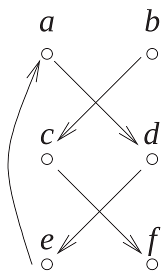
Solutions to Exercise Sheet 7

Exercise 7.1

(a)

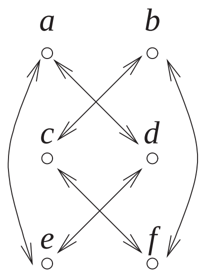


(b) refl-closure$(E) = \{(a,d),(b,c),(c,f),(d,e),(e,a),(a,a),(b,b),(c,c),(d,d),(f,f)\}$.

symm-closure$(E) = \{(a,d),(b,c),(c,f),(d,e),(e,a),(d,a),(c,b),(f,c),(e,d),(a,e)\}$.

trans-closure$(E) = \{(a,d),(b,c),(c,f),(d,e),(e,a),(d,a),(b,f),(e,d),(a,e),(a,a),(d,d),(e,e)\}$

(c) equiv-closure$(E) =$ $\{(a,a)\,,(a,d),(a,e),$
$(b,b),(b,c),(b,f),$
$(c,c),(c,f),(c,b)$
$(d,d),(d,e),(d,a),$
$(e,e),(e,a),(e,d)$
$(f,f),(f,b),(f,c)\}$

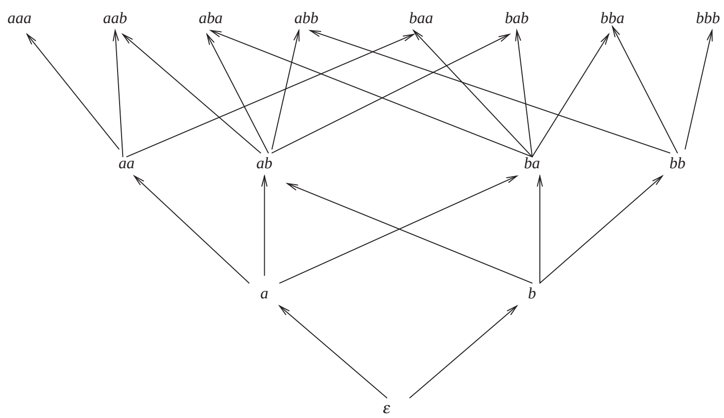


(In the graph I left out out the self-connections)

(d) We get two equivalence classes: $[a]_\approx = [d]_\approx = [e]_\approx = \{a,d,e\}$ and $[b]_\approx = [c]_\approx = [f]_\approx = \{b,c,f\}$

Exercise 7.2

The relation is reflexive because a string is always a substring of itself. It is anti-symmetric because if a string s is a substring of a string t and also the other way around then they must have the same number of characters and in fact be equal. It is also clearly transitive. The picture:



Exercise 7.3

(a) Reflexivity: $a-a$ is equal to zero and zero is divisible by any number (different from zero).

Symmetry: If $b-a$ is divisible by n then so is $a-b$, being just the negative of $b-a$.

Transitivity is more interesting: If we have that both $b-a$ and $c-b$ are divisible by n then we get $c-a = (c-b)+(b-a)$, that is, $c-a$ is the sum of two numbers divisible by n, so is itself divisible by n.

(b) We assume that $b-a$ and $b'-a'$ are divisible by n.

(i) $(b+b')-(a+a') = (b-a)+(b'-a')$ which is the sum of two numbers divisible by n, so is itself divisible by n.

(ii) $bb'-aa' = bb'-ab'+ab'-aa' = (b-a)b'+a(b'-a')$ and if $b-a$ is divisible by n, so is $(b-a)b'$, and similarly for $a(b'-a')$.

(iii) This is best shown by induction over c:
If $c=0$ then $a^0 = 1 = b^0$ and the two expressions are definitely equivalent modulo n.
If the statement is true for some c then it is also true for $c+1$ because $a^{c+1} = a \times a^c$ and $b^{c+1} = b \times b^c$ and we can just apply (ii) to $a \equiv b \pmod{n}$ and $a^c \equiv b^c \pmod{n}$.

(c) $4^6 = 4096$ and $4096-1 = 4095$ is divisible by 7: $4095 = 585 \times 7$.

$4^{10000} = 4^{1666\times 6+4} = (4^6)^{1666} \times 4^4 \equiv 1^{1666} \times 256 = 256 \pmod{7}$ and since $256 = 36\times 7+4$ we have $256 \equiv 4 \pmod{7}$ and the answer is $r = 4$.