The University of Birmingham
School of Computer Science
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Exercise Sheet 7

Exercise 7.1

Consider the set $V = \{a,b,c,d,e,f\}$ and the relation $E = \{(a,d),(b,c),(c,f),(d,e),(e,a)\}$.

(a) Draw a picture of this as a directed graph with vertices V and edges E. [1 point]

(b) Compute (separately) the edge-sets for the reflexive, symmetric, and transitive closure of E (no need to draw the resulting graphs). [3 points]

(c) Compute the edge-set for the equivalence relation generated by E, and draw this as a graph. [2 points]

(d) Write out the classes of the classification which is derived from this equivalence relation. [1 point]

Exercise 7.2

On the set of strings Σ^* over some alphabet Σ consider the binary relation $\triangleleft$ which holds between two strings s and t if s is a substring of t. Spell out why this is an order relation. Draw the order diagram in the style of Box 90 for all strings over the alphabet $\Sigma = \{a,b\}$ (including the empty string ε) up to length 3. [3 points]

Exercise 7.3

[Arithmetic modulo n] Let n be a fixed natural number greater than 1. Say that two integers a and b are *equivalent modulo n* if $b-a$ is divisible by n. This is usually written as $a \equiv b \pmod{n}$.

(a) Check that the three conditions for an equivalence relation are satisfied. [3 points]

(b) Show that if $a \equiv b \pmod{n}$ and $a' \equiv b' \pmod{n}$ then the following is also true:

(i) $a + a' \equiv b + b' \pmod{n}$ [1 point]
(ii) $a \times a' \equiv b \times b' \pmod{n}$ [1 point]
(iii) $a^c \equiv b^c \pmod{n}$ (for any natural number c) [1 point]

(c) Check that $4^6 \equiv 1 \pmod{7}$. [1 point]

Use this and the rules from part (b) to find a number r between 0 and 6 such that $4^{10000} \equiv r \pmod{7}$. [1 point]

Total points: 18

— turn over —

Stretcher Exercise 7

(You can earn two *bonus points* by answering this question. Send your solution via email directly to `O.K.Klinke@cs.bham.ac.uk`.)

[A lot of text but hopefully quite straightforward.] On the handout is a construction of *rationals* out of the set of fractions. It can be argued that there is a similar construction that leads to the *integers* from pairs of natural numbers. Let's go through this step by step:

(a) The base set of individuals is $A = \mathbb{N}\times\mathbb{N}$. Think of the first number as your savings account (money you own), and the second as your mortgage account (money you owe). The idea is that a pair (a,b) represents the "integer" $a-b$. You can also think about this in terms of implementing an "integer" datatype in a programming language that only supports natural numbers (also known as "unsigned integers" in C and C++).

  (i) When would you say that a pair (a,b) represents a positive integer (i.e., you are in credit)?

  (ii) When would you say that a pair (a,b) represents a negative integer (i.e., you are in debt)?

  (iii) When would you say that a pair (a,b) represents zero?

(b) Given two pairs (a,b) and (c,d), when would you say that they represent the same integer? Only use addition in your answer, not subtraction (as the latter could lead to negative numbers which we pretend we don't have yet).

(c) Using only addition and multiplication of natural numbers, define the operations

  (i) addition,

  (ii) subtraction, and

  (iii) multiplication

  on pairs (a,b).

(d) Draw a picture of $\mathbb{N}\times\mathbb{N}$ and indicate which dots belong to the same equivalence class.

(e) For comparison, consider a different implementation of integers that uses a boolean field for the sign (where "true" is interpreted as "+", and "false" as "–"), and a natural number field. Write out the code for a method that implements subtraction of two integers and only uses addition, and subtraction of natural numbers where the first is greater equal the second.