Solutions to Exercise Sheet 9

Exercise 9.1

(a) $2 \in X$ by Rule 1, $2 \times 2 = 4 \in X$ by Rule 3, $3 \in X$ by Rule 2, $3 \times 3 = 9 \in X$ by Rule 3, and $4 \times 9 = 36 \in X$ by Rule 3.

(b) The property I would like to prove by induction is "All elements of X are divisible by 2 or by 3 (or both)".

Base case 1: The statement is true for $x = 2$ as this is divisible by 2.

Base case 2: The statement is true for $x = 3$ as this is divisible by 3.

Inductive step: If x and y are members of X then by induction hypothesis, x is divisible by either 2 or 3, and then this is also true for any multiple of x, so it is true for $x \times y$.

The number $35 = 5 \times 7$ is not divisible by either 2 or 3, so can't be a member of X.

Exercise 9.2

(a) $2 \in X$ by Rule 1, then $2^2 = 4 \in X$ by Rule 2, then $4^2 = 16 \in X$ by Rule 2, then $16^2 = 256 \in X$ by Rule 2 again.

(b) Checking the base case: 2 has the form $2^1 = 2^{2^0}$ so the statement is true with $n = 0$.

Inductive step: Assume $x \in X$ and by induction hypothesis that x has the form 2^{2^n}. Then $x^2 = (2^{2^n})^2 = 2^{2^n \times 2} = 2^{2^{n+1}}$ and the statement is again true with n replaced by $n + 1$.

(c) We use the insight gained in the previous item:

- Base case: Include the pair $(2, 0)$ in the relation.
- Inductive step: If the relation already contains the pair (x, n) then also include the pair $(x^2, n + 1)$.

Exercise 9.3

(a) Re-writing the grammar, we have one base case,

- "`a`" is a valid string in S

and two inductive steps:

- If s and t are from S, then so is bst.
- If s is from S then so is as.

The derivation: `a` $\in S$ by Rule 1, `baa` $\in S$ by Rule 2, `bbaaa` $\in S$ by Rule 2, `abbaaa` $\in S$ by Rule 3, and `aabbaaa` $\in S$ by Rule 3.

(b) This is true for the base case, a. If it is true for s and t then the string st has at least 2 more a's than b's, so bst has at least 1 more a than b's, so the statement remains true. The reasoning for Rule 3 is trivial.

(c) `aba`

(d) I suggest using the grammar:

$$S \quad ::= \quad \mathtt{b}SS \quad | \quad S\mathtt{b}S \quad | \quad SS\mathtt{b} \quad | \quad \mathtt{a}S \quad | \quad \mathtt{a}$$

Exercise 9.4

(a) Obviously, the relation is reflexive as every bracket expression has the same level of nesting as itself. It is also transitive because the order on natural numbers is transitive. It is not anti-symmetric because two different expressions can have the same level of nesting, for example `[[]][]` and `[[]]`.

(b) **base case** $(\texttt{[]}, \texttt{[]})$ belongs to R.

inductive case 1 If (s,t) belongs to R then so does $(\texttt{[}s\texttt{]},t)$.

inductive case 2 If (s,t) belongs to R then so does $(\texttt{[}s\texttt{]},\texttt{[}t\texttt{]})$.

inductive case 3 If (s,t) and (s,t') belong to R then so does (s,tt').

inductive case 4a If (s,t) belongs to R and $s' \in wbb$ then also $(ss',t) \in R$.

inductive case 4b If (s,t) belongs to R and $s' \in wbb$ then also $(s's,t) \in R$.