The University of Birmingham
School of Computer Science
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Exercise Sheet 9

Exercise 9.1

Let X be the smallest set of natural numbers such that the following three conditions are satisfied:

- $2 \in X$;
- $3 \in X$;
- if $x \in X$ and $y \in X$, then $xy \in X$ (where xy is the product of x and y).

(a) Apply the rules repeatedly to demonstrate that 36 is an element of X. [1 point]

(b) Argue that 35 is not an element of X by giving (and proving) that all elements of X satisfy a certain property. [2 points]

Exercise 9.2

Let X be the smallest set of natural numbers such that the following two conditions are satisfied:

- $2 \in X$;
- if $x \in X$ then $x^2 \in X$.

(a) Apply the rules repeatedly to demonstrate that 256 is an element of X. [1 point]

(b) Use structural induction to show that every element of X is of the form 2^{2^n}. [2 points]

(c) Give the (inductive) definition of a function that returns for each element $x = 2^{2^n}$ the number n. [2 points]

Exercise 9.3

Consider the following inductive definition of a set of strings (written as a grammar for brevity):

$$S \quad ::= \quad \mathtt{b}SS \quad | \quad \mathtt{a}S \quad | \quad \mathtt{a}$$

and the string `aabbaaa`.

(a) Find a derivation for this string. [2 points]

(b) Use structural induction to show that every string that can be derived from the grammar will have strictly more `a`'s than `b`'s. [2 points]

(c) Give an example of a string with strictly more `a`'s than `b`'s, which can *not* be generated by the grammar. [1 point]

(d) [Earn 2 bonus points! Send your solution via email to Olaf.] Extend the grammar so that *all* strings with strictly more `a`'s than `b`'s can be generated. Outline the argument why you believe your grammar to satisfy this specification.

Exercise 9.4

[From the May examination 2009] This question is about the inductively defined set *wbb* of "well-balanced (square) brackets," considered in the lecture.

For two elements s and t of *wbb* say that they are in relation R if the level of nesting of s is greater or equal than that of t.

(a) Obviously, R is reflexive and transitive, but it is not anti-symmetric. Give an example that illustrates the lack of anti-symmetry. [1 point]

(b) Show that the relation R (as a subset of $wbb \times wbb$) can itself be defined inductively. [3 points]

Total points: 17

Stretcher Exercise 9

(You can earn two *bonus points* by answering this question. Send your solution via email directly to `O.K.Klinke@cs.bham.ac.uk`.)

Find an inductive definition for the set of strings over the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ that contain as many `a`'s as `b`'s. So the strings ε, "`abba`", and "`bababa`" are members but "`bab`" or "`aabba`" are not. Give a valid argument that proves that all such strings can indeed be derived from your inductive definition.