Propositional Logic and Predicate Logic

Propositional logic
Logical connectives
Rules for wffs
Truth tables for the connectives
Using Truth Tables to evaluate expressions
Equivalences
Formally representing English sentences
Arguments and validity
Rules of deduction

Entailment
Soundness and completeness of inference rules

Predicate logic
More formally representing English sentences
Review of logic from last lecture
Vocabulary of propositional logic

Vocabulary of propositional logic: proposition letters, connectives, parentheses

A **PROPOSITION LETTER** is any symbol from following list:

A, ...Z, A\textsubscript{0}...Z\textsubscript{0}, A\textsubscript{1}...Z\textsubscript{1}...

The **PROPOSITIONAL CONNECTIVES** are \textasciitilde, \lor, \land, \rightarrow, \leftrightarrow

An **EXPRESSION** of propositional logic is any sequence of sentence letters, propositional connectives, or left and right parentheses.

**METAVARIABLES** such as Φ and Ψ are not part of propositional logic, but are used to stand for expressions of propositional logic.
Logical connectives for propositional logic

These connectives correspond to (but are not identical with) words in natural language that serve to connect declarative sentences

¬, ‘it is not the case’,

∨, ‘Either ... or ...’ in its inclusive sense

∧, ‘Both ... and ...’

→, ‘If ... then ...’

↔, ‘If and only if’
Seven rules for forming a wff

(I) ATOMIC SENTENCE, a propositional letter standing on its own is a wff

(ii) NEGATION, if $\Phi$ is a wff, then the expression denoted by $\neg \Phi$ is also a wff

(iii) CONJUNCTION, if $\Phi$ and $\Psi$ are both wffs, then the expression denoted by $(\Phi \land \Psi)$ is a wff

(iv) DISJUNCTION if $\Phi$ and $\Psi$ are both wffs, then the expression denoted by $(\Phi \lor \Psi)$ is a wff

(v) CONDITIONAL (with ANTECEDENT and CONSEQUENT) if $\Phi$ and $\Psi$ are both wffs, then the expression denoted by $(\Phi \rightarrow \Psi)$ is a wff

(vi) BICONDITONAL if $\Phi$ and $\Psi$ are both wffs, then the expression denoted by $(\Phi \leftrightarrow \Psi)$ is a wff

(vii) Nothing else is a wff
Which of these are wffs?

A
A B
(A ∧ B)
(A → B)
(A ↔ B)
(A ∨ B)
(A ¬ B)
A → B ¬ → C
(A → B) → C
(A → ( B)) → C
(A → ∧ B) → C
(A → ¬ B) → C
¬(A → ¬ B) → C
¬(A ¬ B) → C
¬(A → B) → ¬ C
¬(A → B) → ¬ (C ∧ D)
Rules of precedence and parentheses dropping

The order of precedence is \( \neg, \land, \lor, \rightarrow, \leftrightarrow \)

This means that for a formula without brackets that the \( \neg \) is applied first and the \( \leftrightarrow \) is applied last.

Which of the following formulae are ambiguous and which are clear:

- \( A \rightarrow B \rightarrow C \)
- \( (A \rightarrow B) \rightarrow C \)
- \( A \rightarrow B \land C \rightarrow D \)
- \( (A \rightarrow B) \land (C \rightarrow D) \)
From English to Propositional Logic - translation schemes

P: BlockA is small
Q: BlockA is light
The sentence of English: “if BlockA is small then BlockA is light” has the logical form (P → Q)
Stylistic variants in English for logical negation

John is not conscious
John is unconscious
It is not the case that John is conscious
It is false that John is conscious
Stylistic variants in English for logical conditional \((\Phi \to \Psi)\)

If \(\Phi\), then \(\Psi\)
If \(\Phi, \Psi\)
\(\Phi\) is a sufficient condition for \(\Psi\)
\(\Phi\) is sufficient for \(\Psi\)
in case \(\Phi, \Psi\)
Provided that \(\Phi\), then \(\Psi\)
\(\Psi\) provided that \(\Phi\)
\(\Psi\) only on the condition that \(\Phi\)
\(\Psi\) is necessary for \(\Phi\)
\(\Psi\) if \(\Phi\)
Stylistic variants in English for logical conjunction ($\Phi \land \Psi$)

$\Phi$ and $\Psi$
Both $\Phi$ and $\Psi$
$\Phi$, but $\Psi$
$\Phi$, although $\Psi$
$\Phi$ as well as $\Psi$
Though $\Phi$, $\Psi$
$\Phi$, also $\Psi$
Stylistic variants in English for logical disjunction \((\Phi \lor \Psi)\)

\(\Phi \text{ or } \Psi\)
Either \(\Phi \text{ or } \Psi\)
\(\Phi \text{ unless } \Psi\)
Stylistic variants in English for logical biconditionals ($\Phi \leftrightarrow \Psi$)

$\Phi$ if and only if $\Psi$
$\Phi$ is equivalent to $\Psi$
$\Phi$ is necessary and sufficient for $\Psi$
$\Phi$ just in case $\Psi$
Translating from English to Propositional Logic

P: A purpose of punishment is deterrence
Q: Capital punishment is an effective deterrent
R: Capital punishment should be continued
S: Capital punishment is used in the United States
T: A purpose of punishment is retribution

(Allen and Hand page 16)
Translating from English to Propositional Logic

If a purpose of punishment is deterrence and capital punishment is an effective deterrent, then capital punishment should be continued.

Capital punishment is not an effective deterrent although it is used in the United States.

Capital punishment should not be continued if it is not an effective deterrent, unless deterrence is not a purpose of punishment.
Translating from English to Propositional Logic

If a purpose of punishment is deterrence and capital punishment is an effective deterrent, then capital punishment should be continued

\[ P \land Q \rightarrow R \]

Capital punishment is not an effective deterrent although it is used in the United States

\[ \neg Q \land S \]

Capital punishment should not be continued if it is not an effective deterrent, unless deterrence is not a purpose of punishment

\[ (\neg Q \rightarrow \neg R) \lor \neg P \lor (P \rightarrow (\neg Q \rightarrow \neg R)) \]
Computing Truth values

Under any given interpretation atoms have values - *True* or *False*.

Given the values of atoms (or expressions built of those atoms) under some interpretation, *truth tables* can be used to compute a value for any wff under that same interpretation.

We can think of each line in a Truth Table as a Possible World, also called a Model.

Inferences rules can also be used to show that an expression is a provable theorem. First we will consider Truth tables, and then look again at inference rules.

(Russel and Norvig page 203, Nilsson page 222)
Truth Tables for the connectives - conjunction

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A ∧ B</th>
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</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>
Truth Tables for the connectives - disjunction

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A ∨ B</th>
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<tbody>
<tr>
<td>T</td>
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</table>
Truth Tables for the connectives - conditional

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A → B</th>
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<tbody>
<tr>
<td>T</td>
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</table>
## Truth Tables for the connectives - biconditional

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>$A \leftrightarrow B$</th>
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<tbody>
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<td>T</td>
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</tbody>
</table>
Truth Tables for the connectives - negation

<table>
<thead>
<tr>
<th>A</th>
<th>¬ A</th>
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<tbody>
<tr>
<td>T</td>
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</table>
Using Truth Tables to evaluate complex expressions

\[(p \rightarrow q) \land (q \rightarrow p)\]

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<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3 (p \rightarrow q)</th>
<th>4 (q \rightarrow p)</th>
<th>5 ((p \rightarrow q) \land (q \rightarrow p))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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Using Truth Tables to find tautologies

$$((p \to q) \land (q \to p)) \to (p \to q)$$

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<th>3</th>
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<tbody>
<tr>
<td>p</td>
<td>q</td>
<td>(p→q) ∧ (q→p)</td>
<td>(p→q)</td>
<td>(3→4)</td>
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</table>
Using Truth Tables to find tautologies

\[((p \rightarrow q) \land (q \rightarrow p)) \rightarrow (p \rightarrow q)\]

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<td>(p→q) \land (q→p)</td>
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Entailment

Entailment can be demonstrated using truth tables. We can view Possible Worlds (Models) where each atom has either True or False values, and where every possible combination of true and false for all atoms is represented. If one logical sentence follows from another we say that it entails it.

\[ \alpha \models \beta \ (\alpha \text{ entails the sentence } \beta) \]

Formally, \( \alpha \models \beta \) iff, in every model in which \( \alpha \) is true \( \beta \) is also true. Informally, the truth of \( \beta \) is contained in \( \alpha \).

(Russel and Norvig page 201)
What use are tautologies?

If in every possible world (in every model) the sentence is true then it is a valid sentence (tautology). Tautologies are necessarily true and hence vacuous.

The deduction theorem (known to the ancient Greeks):
For any sentences $\alpha$ and $\beta$, $\alpha \models \beta$ if and only if the sentence $(\alpha \rightarrow \beta)$ is valid

Every valid implication sentence describes a legitimate inference, and automating truth table checking over large Knowledge Bases can be thought of as checking inferences, ie that $\text{KB} \rightarrow \alpha$
Valid Inferences and tautologies

P \rightarrow Q
P
therefore Q
can be seen to be a valid inference because the conditional statement

((P \rightarrow Q) \land P) \rightarrow Q

is a tautology

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<td>(P \rightarrow Q) \land P</td>
<td>((P \rightarrow Q) \land P) \rightarrow Q</td>
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Soundness and Completeness of inference rules - informal

Soundness of inference rules is important because we do not want our inference rules to derive theorems that are not logically entailed. The only theorems we want to derive are those that are entailed. Any set of inference rules that can do this is called sound.

If our inference rules can find every logically entailed theorem then those inference rules are said to be complete.
Soundness and Completeness of inference rules - formal

1. For any set of wffs, $\Delta$, and wff $\omega$: if $\Delta \vdash \mathcal{R} \omega$ implies $\Delta \models \omega$ we say that the set of inference rules, $\mathcal{R}$ is sound.

2. For any set of wffs, $\Delta$, and wff $\omega$: if it is the case that whenever $\Delta \models \omega$, there exists a proof of $\omega$ from $\Delta$ using inference rules $\mathcal{R}$ we say that the set of inference rules, $\mathcal{R}$ is complete.

If the rules of inference are sound and complete they can be used to establish a truth value for any wff.

(Nilsson page 222)
Conclusion

Logical connectives, wffs and rules of precedence
Formalising English to Propositional Logic
Truth tables
What we really want is just entailed theorems (no theorems that aren’t entailed)
We get this when our inference rules are sound
we get every entailed theorem that exists when our inference rules are complete
(Do all logics have complete inference rules?)
Next lecture - Deduction in Propositional and FOPL