Hempel's Ravens
How can we test any general theory such as "all ravens are black?"

Natural deduction in Predicate Logic
Since Predicate Logic is an extension of Propositional Logic, all rules for Natural Deduction in PL can be applied in Predicate Logic. We therefore only need to add additional rules for the “extra bits”.

Universal Elimination
1. \( \forall x \ [Fx] \) \( \{n\} \)
2. \( Fa \) \( \text{UE} \) \( \{n\} \)
That is, given a universal rule, we can infer a particular instance.

Universal Introduction
Consider under what conditions we might be able to introduce a universal rule (excluding of course induction). We cannot do the following
1. \( Fa \) \( \{n\} \)
2. \( \forall x \ [Fx] \) \( \{n\} \)
Since this is a form of induction (and recall Hempel's ravens for problems with such things!)

However we can do the following:
1. \( \forall x \ [Fx \land Gx] \) \( \text{Premise} \) \( \{1\} \)
2. \( Fa \land Ga \) \( \text{UE} \) \( \{1\} \)
3. \( Fa \land E \) \( \{1\} \)
4. \( \forall x \ [Fx] \) \( \text{UI} \) \( \{1\} \)
That is, as long as our introduction of a Universal is based on another Universal involving the same variables then we can introduce a new universal.

There is another possible situation where a universal can be added. Consider:
1. \( Fc \) \( \{1\} \)
2. \( \forall x \ [Fc] \) \( \text{UI} \) \( \{1\} \)
More generally, UI may not be applied to any formula containing a name which includes among its dependencies any formula which itself contains that name.

Existential Introduction
Consider
1. \( Fa \) \( \{1\} \)
2. \( \exists x \ [Fx] \) \( \text{EI} \) \( \{1\} \)
That is, if \( Fa \) is true then there's at least one \( x \) such that \( Fx \). This should seem obvious. However it's caused a fair amount of controversy. Consider:
1. \( \forall x \ [Fx] \) \( \text{Premise} \)
2. \( Fa \) \( \text{UE} \) \( \{1\} \)
3. \( \exists x \ [Fx] \) \( \text{EI} \) \( \{2\} \)

However, we previously said that a universal rule does not assume existence! I will provide some reading on this topic for the interested. For now, let's just note that this has generated much heated
debate in logic. (My personal view is that propositions such as \( \forall x \ [Fx] \) should be viewed with suspicion. Notice that \( \forall x \ [Fx \rightarrow Zx] \) doesn't allow \( Fa \) to poof into existence - only something which if it had F-ness then it would have Z-ness).

**Existential Elimination**
This is probably the hardest rule to understand. It's best to imagine it as a kind of disjunction elimination. Consider:

1. \( \exists x \ [Fx] \) \{1\}
Consider all the possible things in our domain (or the Universe)
therefore \( Fa \lor Fb \lorFc \lorFd \ldots \lorFz \) etc.
2. \( | \ Fa \ H \)
3. \( | \ B \)
4. \( B \ EE \ \{1,2\} \)
That is, given an Existential, we can assume a “typical” member of the disjunction, and if this allows us to prove “B” then we can eliminate the existential.

Clearly we need to place restrictions on what we choose to be a “typical” member. In particular, we must ensure that the name in the “typical” disjunct is **not** contained in the conclusion B or that the name contained in the “typical” disjunct is contained in any premise or assumption used to derive the conclusion for EE.

This is typically a hard rule to understand. I'll cover this more next time. However consider how EE can work with EI ...

**A short example of the new Natural Deduction rules in action**
\( \forall x \ [Fx \rightarrow Gx], \ Fa : \exists x \ [Gx] \)

1. \( \forall x \ [Fx \rightarrow Gx] \) Premise \{1\}
2. \( Fa \) Premise \{2\}
3. \( Fa \rightarrow Ga \ UE \ \{1\} \)
4. \( Ga \rightarrow I \ \{1,2\} \)
5. \( \exists x \ [Gx] \ El \ \{1,2\} \)
We'll do more of this during the classwork.
Identity
Consider the following snippet of predicate logic.
\[
\begin{align*}
I & \{\text{Europeans}\} \\
F &= \ldots \text{is a French Man} \\
G &= \ldots \text{is a German} \\
P &= \ldots \text{is the President of France} \\
s &= \text{Sarkozy} \\
h &= \text{Hans} \\
Fs \land Ps
\end{align*}
\]
That is “Sarkozy is a French man and Sarkozy is the President of France”. Suppose (and this might be quite a scandal) that in fact Nicolas Sarkozy was a German spy called “Hans” How could we represent this in Predicate Logic?
We need to add the concept of “identity” such that \( h = p \)
Note that this is different from making a predicate such as “H = is hans” since we are dealing with two named individuals. In fact identity greatly increases the expressive power of our formal language.

The law of identity
\[
\begin{align*}
\forall x \ [x = x]
\end{align*}
\]
This should be obvious. More interesting (perhaps) is that fact that identity can be used to represent non-equivalence i.e.
\[
\neg(h = s)
\]
i.e. Hans & Sarkozy are not the same person. Before we introduced identity, this fact could not be represented in predicate logic.
A second use of identity is that it allows us to be numerically definite about existence. Consider how we might represent:

There is one and only one French President.

An obvious (but partially wrong) answer is
\[
\exists x \ [Px]
\]
Since this states that there exists one (and possibly more) French Presidents. However, we can use identity. Given a domain, consider that if anything x has the property P then if y also has the property P then x = y. i.e.
\[
\forall x \ [\forall y \ [(Px \land Py) \rightarrow (x = y)]]
\]
Therefore, if we wish to say that there is exactly one Px then we need to use the conjunction of both formulae i.e.
\[
\exists x \ [Px] \land \forall x \ [\forall y \ [(Px \land Py) \rightarrow (x = y)]]
\]
We can extend this to cover any number of discrete entities by the use of \( \neg(x=y) \) i.e. For two entities:
\[
\exists x \ [\exists y \[((Fx \land Fy) \land \neg(x = y)) \land \forall z \ [Fz \rightarrow ((z = x) \lor (z = y))]\]]
\]
Of course now we can represent numbers, we can capture mathematics within logic ...
(but see lecture 10)