Further Inference In Predicate Logic

Dr. Mark Lee
mgl@cs.bham.ac.uk

The Surprise Quiz

A logic teacher informed his class that there would be a surprise logic quiz in the last three weeks of term. The brightest student immediately put his hand up and objected that such a thing was impossible: *"Will the quiz be given during during the third week? If it were then the quiz wouldn't have taken place in weeks one and two. At the end of the second week, we'd know the quiz had to be in the third week but this cannot be because it's a "surprise quiz". So will the quiz happen during the second week? We already know it cannot happen during the third week and so at the end of the quiz-less first week we'll know that the exam is in the second week but this cannot be because it's a "surprise quiz". The only remaining possibility is that the quiz is in the first week. But we know this therefore this cannot be a "surprise quiz". Therefore a surprise quiz is impossible."*

The Non-surprise quiz

Last Week of Term Monday 4-4.45 am 5th of December.

Expect to be tested on everything we've covered in this module.

Natural Deduction and Identity

Last week we introduced identity into our notation for predicate logic. As a recap consider

∃x [Fx & ~(x = m)] or
∀x [Fx -> (x = m)]

where m is some named entity. If we want to use infer things involving identity then we need to add rules to our set of natural deduction rules. Fortunately the rules are rather easy to understand (and use).

Identity Introduction

1. a = a =I

That is, we can introduce the identity relation for any name at any stage of a proof.

Identity Elimination

1. a = b	Premise	{1}
2. Fa	Premise	{2}
3. Fb	=E	{1,2}

That is, given an identity relation and a fact involving one name in the identity we can infer a new fact with the other name. Similarly :

1. a = b	Premise	{1}
2. b = c	Premise	{2}
3. a = c	=E	{1,2}

i.e. Identity has the property of transitivity.

A quick example of the rules in action

: ∀x [∃y [x = y]]

1. x = x	=I {1}	
2. ∃y [x = y]	EI {1}	
3. ∀x [∃y [x = y]]	UI {1}	

Identity & Negation
The methods of negation introduction are useful for proving ~(a = b) i.e. Proving that that two names refer to different entities. Consider:

Fa, ~Bb, Fa -> Ba : ~(b = a)

1. | b = a	H	{1}
2. | Fa	Premise	{1,2}
3. | Fa -> Ba	Premise	{1,3}
4. | Ba	->E	{1,2,3}
5. | ~Bb	Premise	{1,5}
6. | Bb	=E	{1,2,3}
7. ~(b = a)	RAA	{2,3,5}

Strategies for Natural Deduction Proofs in Predicate Logic

Recall the three rules of thumb we covered for PL

1. If the conclusion is a conditional then assume the antecedent and attempt to prove the conclusion. If you can then use the conditional introduction rule to prove the conclusion.

2. If the premises contain a disjunction then attempt to use the disjunction elimination rule.

3. If neither of the above apply then attempt to use Reductio Ad Absurdium as a strategy i.e. Assume the opposite of what you're trying to prove and then show a contradiction then apply RAA to prove the conclusion must therefore be true.

With the addition of the Universal and Existential Quantifiers, we can add to these rules of thumb as follows:

If the premises consist of just universals and unquantified formulae and the conclusion is unquantified.

Apply UE to each universal premise to infer formula(s) about the individual(s) named in the unquantified individuals. Then apply the golden rule stated above to infer the conclusion.

∀x [Dx -> Bx], Da: Ba

1. ∀x [Dx -> Bx]	Premise {1}
2. Da -> Ba	UE {1}
3. Da	Premise {3}
4. Ba	->E {1,3}

If the premises consist of universals & unquantified formulae and the conclusion is an existential

Apply UE to each universal premise to infer formulas about the individuals named in the unquantified formulas. Apply the golden rule above to infer the desired conclusion about some named individual and then apply EI to infer the desired existentially quantified conclusion.

∀x [Dx -> Bx], Da: ∃y [By]

...

5. ∃y [By] EI {1,3}

If the premises are all universal and the conclusion is universal

Apply UE to each premise to infer formulas about one particular individual (i.e. Infer formulas about one name – say “a”) then apply the golden rule to yield the desired conclusion for the named individual. Make sure all assumptions are discharged in the sub-proof. Apply UI to complete the proof.

∀x [Dx v Bx], ∀x [Dx -> Bx]: ∀x [Bx]

1. ∀x [Dx v Bx]	Premise {1}
2. Da v Ba	UE1 {1}
3. ∀x [Dx -> Bx]	Premise {3}
4. Da -> Ba	UE3 {3}
5. | Da	H {5}
6. | Da -> Ba	i 4 {3,5}
7. | Ba	->E 5,6 {3,5}
7. | Ba	H {7}
8. | Ba	i 7 {7}
10 Ba	VE 5,6,7,8,9 {1,3}
11 ∀x [Bx]	UI 10 {1,3}

"EI before EE!"

In cases where the premises contain an existential and conclusion is existentially quantified, a general rule is "EI before EE". We need to be very careful with this and I expect this might take a bit of work to understand. First let's deal with premises.

If the premises are all existential ...

1. Construct an argument from a typical disjunct i.e. Carefully select and assume a disjunct which is genuinely typical in the context of the proof.

2. Apply the golden rule to determine the correct strategy for deriving the desired conclusion from the formula assumed as typical disjunct

...

If the premises are universal and existential and the conclusion existential

1. First carefully select and assume genuinely typical disjuncts. Then apply UE to the remaining premises to infer formulas about the same individuals.

2. Apply the golden rule to determine the correct strategy for sub-proof

...

Once we've done the sub-proof we can worry about the conclusion

If the conclusion is existentially quantified

3. Apply EI before EE and if necessary reapply the golden rule

4. Check the restrictions on EE are met and then apply EE to complete the proof.

Tomassi (pages 315-320) covers the general strategies in greater detail than these notes though some parts of his explanation I personally found rather hard to follow.

A very short (if unsatisfying) example.

∃x [Fx] : ∃y [Fy]

1. ∃x [Fx]	Premise {1}
2. | Fa	H {2}
3. | ∃y [Fy]	EI {2}
4. ∃y [Fy]	EE {1}

A long and more involved example

∀y [Gy -> Hy] : ∃x [Gx]-> ∃z [Hz]

1 | ∃x [Gx]	H {1}
2 | | Ga	H {1,2}
3 | | ∀y [Gy -> Hy]	Premise {1,2,3}
4 | | Ga -> Ha	UE {1,2,3}
5 | | Ha	->E {1,2,3}
6 | | ∃z [Hz]	EI {1,2,3}
7 | ∃z [Hz]	EE {1,3}
8 ∃x [Gx]-> ∃z [Hz]	->I 1,7 {3}

Notice the same pattern of EI before EE. We'll practice Natural Deduction in Predicate Logic on Thursday.