Compact types and ordinals in constructive univalent type theory

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It is more or less well-known how continuity interacts with constructivity

1. It is less well-known how compactness interacts with constructivity.

2. We investigate this from the point of view of univalent type theory.

I am going to talk about something for which there aren't good models yet

1. I am going to investigate a synthetic notion of compactness, relying on univalent concepts, ideas, and axioms.

2. The synthetic notion is intended to capture that of topology.

3. But the examples we develop are discrete, infinite and hence non-compact in the *simplicial set model*.

4. In the topological topos we get the right interpretation of compactness, but not of the identity types.

Two interpretations of types as spaces

1. - Types are spaces.
   - Functions are continuous.

   Kleene–Kreisel spaces, Scott domains, ... (simple types).

   Johnstone's topological topos (extends this to dependent types).

   All this goes back to Brouwer.

2. - Types are spaces up to homotopy equivalence.

     Types are homotopy types (hence "homotopy type theory").
   - Functions are continuous.

   Voevodsky's simplicial set model.

Can (and should) they be reconciled?

We look at type-theoretical manifestations of the topological notion of compactness to illustrate the point.

I also formulate more precise questions in this direction.

Compactness versus the principle of omniscience

1. Compactness in topology.

   A space X is compact if no matter how we cover it with open sets U_i, finitely many of them already cover it.

   $$X \subseteq \bigcup_{i \in I} U_i \implies \exists J \subseteq I \text{ finite with } X \subseteq \bigcup_{j \in J} U_j.$$

2. Omniscience in constructive mathematics.

   A set X satisfies the principle of omniscience if every detachable subset is either inhabited or empty.

   $$\forall (p : X \to 2).(\exists (x : X).p(x) = 1) \vee (\forall (x : X).p(x) = 0).$$

   Under Curry–Howard, this amounts to a selection function

   $$2^X \to X + 1.$$

   Find an element of the subset, or determine that it is empty.

Higher-type recursion theory gives a frame of comparison

1. The Kleene–Kreisel spaces (1950's) interpret the simple types $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, ...

   Can be calculated as exponentials in the cartesian closed category of *compactly generated spaces* (Martin Hyland, 1970's).

2. The continuous maps between them are known as the Kleene–Kreisel functionals, or simply as the continuous functionals.

3. They interpret the simply typed lambda-calculus with natural numbers, or Gödel's system T (and various extensions of T).

Compactness = continuous omniscience

Theorem (E. 2008). *A subspace of a Kleene–Kreisel space is compact* $\iff$ *it has a continuous selection function.*

Theorem (E. 2008). *T.F.A.E. for a non-empty subspace $X \subseteq K$ of a Kleene–Kreisel space:*

1. X is compact.
2. X has a continuous emptiness decision map $2^X \to 2$.
3. X has a continuous selection function $2^X \to X + 1$.
4. X is a continuous image of the Cantor space $2^{\mathbb{N}}$.

Moreover,

(i) In this case, X is a retract of K.

(ii) The equivalences (2)-(4) are continuous.

For instance, we have a map corresponding to $(2) \implies (3)$,

$$(2^X \to 2) \to (2^X \to X + 1),$$

that continuously transforms an emptiness decision map into a selection function.

Somehow "magically", the knowledge of non-emptiness without a witness allows us to continuously find a witness.

Remark

Let X be a compactly generated space.

Let $\mathbb{S}$ be the Sierpinski space with an isolated point $\top$ and a limit point $\bot$.

1. The exponential 2^X represents clopen sets of X as a space.
2. The exponential $\mathbb{S}^X$ represents the open sets of X as a space (and it has the Scott topology).
3. X is compact $\iff$ the map $\forall_X : \mathbb{S}^X \to \mathbb{S}$ is continuous.
4. It is an oddity of the subspaces of the Kleene–Kreisel spaces that X is compact $\iff$ the map $\forall_X : 2^X \to 2$ is continuous.

   Among other things, the KK spaces are *totally separated*, which means that the clopens separate the points (but not zero dimensional in general).

Problem. Define and study a Sierpinski type in univalent type theory before we consider any model.

We want to transfer this theorem to computability theory

Theorem (E. 2008). *T.F.A.E. for a non-empty subspace X of a Kleene–Kreisel space:*

1. X is compact.
2. X has a continuous emptiness decision map $2^X \to 2$.
3. X has a continuous selection function $2^X \to X + 1$.
4. X is a continuous image of the Cantor space $2^{\mathbb{N}}$.

Replacing *continuous* by *computable*, we get

Theorem (E. 2008). *The following are computably equivalent for a non-empty subspace $X \subseteq K$ of a Kleene–Kreisel space:*

1. X has a computable emptiness decision procedure $2^X \to 2$.
2. X has a computable selection function $2^X \to X + 1$.
3. X is a computable image of the Cantor space $2^{\mathbb{N}}$.

Moreover, in this case, computably in (any of) the above data,

(i) X is a computable retract of K,

(The space of fixed-points of a computable idempotent $K \to K$.)

(ii) membership of a point of K in X is co-semi-decidable.

Computable countable Tychonoff Theorem

Theorem. *Pointed, computably searchable spaces are closed under the formation of countable products.*

1. A selection function for a pointed space X can be given the form

$$2^X \to X.$$

   We answer the given point of X if we find nothing.

   Conversely, we can find a point of X from such a selection function.

2. Computable "Tychonoff" functional

$$\left(\prod_n \left(2^{X_n} \to X_n\right)\right) \to \left(2^{\prod_n X_n} \to \prod_n X_n\right).$$

   Given selection functions for countably many pointed spaces, we get one selection function for their product.

   Pointedness is crucial - without it continuity is violated.

3. Example. 2 is searchable, and hence so is the cantor space $2^{\mathbb{N}}$.

We now turn to type theory

- We begin with Gödel's system T
  (which of course *doesn't* define all computable functions).
- Consider $2^{\mathbb{N}}$ with the lexicographic order.
- Recall that ϵ_0 is the first fixed point of the exponential function $\omega^{(-)}$.

Theorem (E. 2011) *For every ordinal $\alpha < \epsilon_0$, there is an ordinal $\beta < \epsilon_0$ above α and a subset $X \subseteq 2^{\mathbb{N}}$ of order type $[0, \beta]$ such that*

1. X is a Gödel's T definable retract of $\mathbb{N}^{\mathbb{N}}$.

   (The space of fixed-points of a T-definable map $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.)

2. X has a T-definable selection function

   (which chooses minimal elements).

There are T-searchable spaces of Cantor–Bendixon rank arbitrarily close to ϵ_0 from below

Thus, perhaps surprisingly, higher-type primitive recursion allows exhaustive search of plenty of infinite spaces.

Conjecture (E. 2011.) Although there are uncountable, computably searchable spaces such as $2^{\mathbb{N}}$,

1. every T-searchable space must be countable, and, moreover,
2. has to have Cantor–Bendixon rank $< \epsilon_0$.

Dag Normann (2012) settled the conjecture positively.

The above are meta-theorems

1. We have a model, the Kleene–Kreisel spaces.

2. We ask what the computably searchable subspaces look like.

   They are precisely the computable images of the Cantor space $2^{\mathbb{N}}$.

3. We ask what the Gödel's T-searchable subspaces look like.

   They are compact Hausdorff spaces with countably many points.

   They all have Cantor–Bendixon rank arbitrarily close to ϵ_0, but strictly smaller than ϵ_0.

These results are formulated and proved outside system T, or any such type-theoretic formalism, some of them using classical logic.

We can ask the same kind of questions for dependent types

1. The KK spaces fully embed in Johnstone's topological topos.

2. This gives a model of Martin–Löf type theory (MLTT).

   Although also built out of spaces, very different from HoTT models.

   We'll return to that.

3. What can we say about the MLTT-searchable spaces?

But this is not what I want to talk about today, although I will formulate some conjectures and questions later on.

What types can be shown to be searchable within a constructive univalent type theory?

1. At the moment, I only can search "countable" types.

   For a precise, constructively weak, notion of countability.

   But one can get ordinals much higher than ϵ_0.

   How high one can get depends on whether one has W types, on how many universes one has, and more generally which features of MLTT (or HoTT) one incorporates.

2. Of course, adding excluded middle and choice would make this much easier.

   But searchability in the way that I am going to define actually gives global choice, which is in general incompatible with univalence.

3. Conjecture. Again, like in Gödel's system T, we can only get countable searchable types in such a univalent type theory.

   One can, however, extend MLTT with Brouwerian continuity axioms so that the Cantor space $2^{\mathbb{N}}$ becomes searchable.

The “synthetic” notion of compactness we consider is that of having a selection function

We now turn to a constructive development within type theory, without considering any model.

At some points we crucially rely on univalent concepts and axioms.

LPO (Bishop's Limited Principle of Omniscience)

For any given $p : \mathbb{N} \to 2$, we can either find $n : \mathbb{N}$ with $p(n) = 0$, or else determine that $p(n) = 1$ for all $n : \mathbb{N}$.

$$\Pi(p : \mathbb{N} \to 2).(\Sigma(n : \mathbb{N}).p(n) = 0) + (\Pi(n : \mathbb{N}).p(n) = 1)$$

For any given $p : \mathbb{N} \to 2$, we can either find a root of p, or else determine that there is none.

$$\Pi(p : \mathbb{N} \to 2).(\Sigma(n : \mathbb{N}).p(n) = 0) + \neg\Sigma(n : \mathbb{N}).p(n) = 0.$$

Subsingleton version of LPO

Any $p : \mathbb{N} \to 2$ either has a root or it doesn't.

$$\Pi(p : \mathbb{N} \to 2). \|\Sigma(n : \mathbb{N}).p(n) = 0\| + \neg\Sigma(n : \mathbb{N}).p(n) = 0.$$

No need to singleton-truncate the rightmost Σ, as the negation of a type is automatically a subsingleton.

Also, this truncation is definable in MLTT (by considering the existence of a minimal root).

The LPO types

$$\Pi(p : \mathbb{N} \to 2).(\Sigma(n : \mathbb{N}).p(n) = 0) + \neg\Sigma(n : \mathbb{N}).p(n) = 0$$

and

$$\Pi(p : \mathbb{N} \to 2).\|\Sigma(n : \mathbb{N}).p(n) = 0\| + \neg\Sigma(n : \mathbb{N}).p(n) = 0$$

are logically equivalent, but not necessarily homotopy equivalent.

The second is a retract of the first.

(This doesn't use the HoTT formulation of the axiom of choice.)

(It is an instance of choice that just holds.)

LPO is undecided

$$\Pi(p : \mathbb{N} \to 2).(\Sigma(n : \mathbb{N}).p(n) = 0) + \neg\Sigma(n : \mathbb{N}).p(n) = 0$$

1. A meta-theorem is that MLTT doesn't inhabit LPO or ¬LPO.
2. Each of them is consistent with MLTT.

   Classical models validate LPO.

   Effective and continuous models validate ¬LPO.
3. LPO is undecided, and we'll keep it that way.
4. But we'll say it is a constructive taboo.

We now make $\mathbb{N}$ larger by adding a point at infinity

Let $\mathbb{N}_\infty$ be the type of decreasing binary sequences.

$$\mathbb{N}_\infty \stackrel{\text{def}}{=} \Sigma(\alpha : 2^{\mathbb{N}}).\Pi(n : \mathbb{N}).\alpha(n) = 0 \to \alpha(n+1) = 0.$$

Side-remark:

1. $\mathbb{N}$ is the *initial algebra* of the functor $1 + (-)$.
2. $\mathbb{N}_\infty$ is the *final coalgebra* of this functor.

(This requires function extensionality.)

$$\mathbb{N}_\infty \stackrel{\text{def}}{=} \Sigma(\alpha : 2^{\mathbb{N}}).\Pi(n : \mathbb{N}).\alpha(n) = 0 \to \alpha(n+1) = 0.$$

1. The type $\mathbb{N}$ embeds into $\mathbb{N}_\infty$ by mapping the number $n : \mathbb{N}$ to the sequence $\underline{n} \stackrel{\text{def}}{=} 1^n 0^\omega$.
2. A point not in the image of this is $\infty \stackrel{\text{def}}{=} 1^\omega$.
3. The assertion that every point of $\mathbb{N}_\infty$ is of one of these two forms is equivalent to LPO.
4. What is true is that no point of $\mathbb{N}_\infty$ is different from all points of these two forms.
5. The embedding $\mathbb{N} + 1 \to \mathbb{N}_\infty$ is an equivalence iff LPO holds.
6. But the complement of its image is empty. We say it is dense.

Remark

This is analogous to what happens to the universe Prop of subsingletons.

1. We have an embedding $2 \to \mathrm{Prop}$.

   This is an equivalence iff excluded middle holds.

2. But its image always has empty complement.
   (Assuming propositional univalence.)

3. This means that there is no truth-value other than the empty type 0 or the contractible type 1.

4. Or, equivalently, that for any proposition P we cannot simultaneously have $\neg P$ (that is, $P \neq 1$) and $\neg\neg P$ (that is, $P \neq 0$).

5. Hence Prop satisfies the principle of omniscience.

   Given $p : \mathrm{Prop} \to 2$, it suffices to check $p(0)$ and $p(1)$.

   Even though we can't claim in general that for any $P : \mathrm{Prop}$, either $P = 0$ or $P = 1$.

Theorem (E. 2011)

$$\Pi(p : \mathbb{N}_\infty \to 2).(\Sigma(x : \mathbb{N}_\infty).p(x) = 0) + \neg\Sigma(x : \mathbb{N}_\infty).p(n) = 0.$$

This is LPO with $\mathbb{N}$ replaced by $\mathbb{N}_\infty$.

While LPO is undecided and a constructive taboo, the above just holds.

WLPO is also undecided by MLTT

$$\Pi(p : \mathbb{N} \to 2).(\Pi(n : \mathbb{N}).p(n) = 1) + \neg\Pi(x : \mathbb{N}).p(n) = 1$$

(This implies that every Turing machine carries on for ever or it doesn't.)

But we have:

Theorem (E. 2013)

$\Pi(p : \mathbb{N}_\infty \to 2).(\Pi(n : \mathbb{N}).p(\underline{n}) = 1) + \neg\Pi(n : \mathbb{N}).p(\underline{n}) = 1.$

The point is that now we quantify over $\mathbb{N}$, although the function p is defined on $\mathbb{N}_\infty$.

Some consequences

1. Every function $f : \mathbb{N}_\infty \to \mathbb{N}$ is constant or not.
2. Any two functions $f, g : \mathbb{N}_\infty \to \mathbb{N}$ are equal or not.
3. Any function $f : \mathbb{N}_\infty \to \mathbb{N}$ has a minimum value, and it is possible to find the minimal point at which the minimum value is attained.
4. For any function $f : \mathbb{N}_\infty \to \mathbb{N}$ there is a point $x : \mathbb{N}_\infty$ such that if f has a maximum value, the maximum value is x.
5. Any function $f : \mathbb{N}_\infty \to \mathbb{N}$ is not continuous, or not-not continuous (E. 2014).
6. WLPO holds iff there is some non-continuous function $f : \mathbb{N}_\infty \to \mathbb{N}$ (E. 2014).

Are there more types like $\mathbb{N}_\infty$?

1. Plenty.

2. Our business here is how to construct them.

3. Their precise nature is open, however.

Two notions

Definition (Omniscient type)

A type X is omniscient if for every $p : X \to 2$, the assertion that we can find a root of p is decidable.

$$\Pi(p : X \to 2).(\Sigma(x : X).p(x) = 0) + \neg\Sigma(x : X).p(x) = 0.$$

Definition (Searchable type)

A type X is searchable if for every $p : X \to 2$ we can find $x_0 : X$ such that if $p(x_0) = 1$, then $p(x) = 1$ for all $x : X$.

$$\Pi(p : X \to 2).\Sigma(x_0 : X).p(x_0) = 1 \to \Pi(x : X).p(x) = 1.$$

We say that x_0 is a *universal witness* for p.

Their relationship

$$\text{omniscient}(X) \stackrel{\text{def}}{=} \Pi(p : X \to 2).(\Sigma(x : X).p(x) = 0) + \neg\Sigma(x : X).p(x) = 0.$$

$$\text{searchable}(X) \stackrel{\text{def}}{=} \Pi(p : X \to 2).\Sigma(x_0 : X).p(x_0) = 1 \to \Pi(x : X).p(x) = 1.$$

NB. These types are not subsingletons in general.

Proposition. *A type X is searchable iff it has a point and is omniscient:*

$$\text{searchable}(X) \iff X \times \text{omniscient}(X).$$

A few theorems rely on pointedness, using the notion of searchability.

Closure under images

Theorem. *Searchable types and omniscient types are closed under the formation of images.*

If X is searchable/omniscient, then so is the image of f for any Y and any $f : X \to Y$.

Closure under Σ

Theorem. *Searchable types and omniscient types are closed under* Σ.

That is, if

1. X is omniscient/searchable and
2. Y is an X-indexed family of omniscient/searchable types,

then so is its disjoint sum $\Sigma(x : X).Y(x)$.

Closure under Π

Not to be expected in general.

E.g. $\mathbb{N}_\infty$ and 2 are omniscient, but in continuous (and also in effective) models of type theory, the function type $2^{\mathbb{N}_\infty}$ is not.

In the topological topos, the exponential $2^{\mathbb{N}_\infty}$ is a countable discrete space, and hence non-compact.

Closure under finite products

Theorem. *A product of searchable types indexed by a finite type is again searchable.*

Brouwerian closure under countable products

Theorem. Brouwerian intuitionistic axioms $\implies$
A countable product of searchable types is searchable.

This is a kind of Tychonoff theorem, if we think of searchability as a "synthetic" notion of compactness.

In particular, the Cantor type $2^{\mathbb{N}}$, which is interpreted as the Cantor space in the topological topos, is searchable.

1. Falsified in one effective model
   (the effective topos, which is realizability over Kleene's K_1).

2. But validated in another effective model
   (realizability over Kleene's K_2),
   and in the topological topos.

   (I implemented this in Agda by disabling the termination checker in a particular function. One can run interesting examples.)

We will need this form of closure under Π

Theorem (micro Tychonoff).

A product of searchable types indexed by a subsingleton type is itself searchable.

That is, if X is a subsingleton, and Y is an X-indexed family of searchable types, then the type $\Pi(x : X).Y(x)$ is searchable.

This is easy with excluded middle, but we are not assuming it.

This cannot be proved if searchability is replaced by omniscience (that is, if we don't assume that every $Y(x)$ is pointed).

Corollary. *A product of searchable types indexed by a subtype of a finite type is searchable.*

Theorem. A subsingleton-indexed product of searchable types is searchable.

Proof.

1. Let X subsingleton, $Y(x)$ searchable for every $x : X$.
2. $Z \stackrel{\text{def}}{=} \Pi(x : X).Y(x)$.

   We have $\Pi(x : X).(Z \simeq Y(x))$ and $(X \to 0) \to (Z \simeq 1)$.
3. Let $p : Z \to 2$.
4. Construct $z_0(x) \stackrel{\text{def}}{=} \ldots$ in Z using the first equivalence.
5. $X \to p(z_0) = 1 \to \Pi(z : Z).p(z) = 1$.

   $p(z_0) = 1 \to \Pi(z : Z).X \to p(z) = 1$.

   $p(z_0) = 1 \to \Pi(z : Z).p(z) = 0 \to (X \to 0)$.
6. $(X \to 0) \to p(z_0) = 1 \to \Pi(z : Z).p(z) = 1$.

   $p(z_0) = 1 \to \Pi(z : Z).(X \to 0) \to p(z) = 1$.
7. By transitivity of $\to$, we get

   $p(z_0) = 1 \to \Pi(z : Z).p(z) = 0 \to p(z) = 1$, so

   $p(z_0) = 1 \to \Pi(z : Z).p(z) = 1$. Q.E.D.

Amusing consequence, tangential to our development

Consider the subsingleton version of LPO discussed above.

Corollary. *The type $\mathbb{N}^{LPO}$ is searchable.*

- The reason is that LPO implies that $\mathbb{N}$ is searchable, and so this is a product of searchable types.

  Even though the searchability of $\mathbb{N}$ is undecided!

- If LPO holds, the type of the corollary is $\mathbb{N}$.
- If LPO fails, it is the contractible type 1.
- As LPO is undecided, we don't know what the type $\mathbb{N}^{\mathsf{LPO}}$ "really is".
- Whatever it is, however, it is always searchable.

Disjoint sum with a point at infinity

Theorem.

The disjoint sum of a countable family of searchable sets with a point at infinity is searchable.

- We need to say how we add a point at infinity.
- The type $1 + \Sigma(n : \mathbb{N}).X(n)$ won't do, of course.
- We will do this in a couple of steps.

Injectivity of the universe of types

Theorem.
For any embedding $e : A \to B$*, every* $X : A \to U$ *extends to some* $Y : B \to U$ *along* e*, up to equivalence: for all* $a : A$*,*

$$Y(e(a)) \simeq X(a).$$

Definition. A map $e : A \to B$ is called an embedding iff its fibers $e^{-1}(b)$,

$$\Sigma(a : A).e(a) = b,$$

are all subsingletons.

Injectivity of the universe of types

Theorem.
For any embedding $e : A \to B$*, every* $X : A \to U$ *extends to some* $Y : B \to U$ *along* e*, up to equivalence.*

Two constructions:

1. We have the "maximal" extension $Y = X/e$.

$$\begin{aligned}(X/e)(b) &= \Pi\left(s : e^{-1}(b)\right).X(\mathrm{pr}_1\, s) \\ &\simeq \Pi(a : A).\ e(a) = b \ \to\ X(a).\end{aligned}$$

2. And also the "minimal" extension $Y = X \setminus e$.

$$\begin{aligned}(X \setminus e)(b) &= \Sigma\left(s : e^{-1}(b)\right).X(\mathrm{pr}_1\, s) \\ &\simeq \Sigma(a : A).\ e(a) = b \ \times\ X(a).\end{aligned}$$

The first one has further properties that are crucial for our purposes.

Injectivity of the universe of types

Let $e : A \to B$ be an embedding and $X : A \to U$.

Consider the extended type family $X/e : B \to U$ defined above:

$$(X/e)(b) = \Pi\left(s : e^{-1}(b)\right).X(\mathrm{pr}_1\, s)$$

We have

1. For all $b : B$ *not* in the image of the embedding,

$$(X/e)(b) \simeq 1.$$

2. If for all $a : A$, the type $X(a)$ is searchable, then for all $b : B$ the type $(X/e)(b)$ is searchable too, by micro-Tychonoff.

3. Hence if additionally B is searchable, the type $\Sigma(b : B).(X/e)(b)$ is searchable too.

We are interested in $A = \mathbb{N}$ and $B = \mathbb{N}_\infty$, which gives the disjoint sum of $X(a)$ with a point at infinity.

A map $L : (\mathbb{N} \to U) \to U$

1. Let $e : \mathbb{N} \to \mathbb{N}_\infty$ be the natural embedding.
2. Given $X : \mathbb{N} \to U$, first take $X/e : \mathbb{N}_\infty \to U$.
3. This step adds a point at infinity to the sequence.
4. We then sum over $\mathbb{N}_\infty$, to get $L(X)$:

$$L(X) = \Sigma(u : \mathbb{N}_\infty).(X/e)(u).$$

By the above development, L maps any sequence of searchable types to a searchable type.

Iterating this map $L : (\mathbb{N} \to U) \to U$

By iterating this, we get (very large!) searchable ordinals, with the property that any non-empty *decidable* subset has a least element.

They are all countable.

Or rather each of them has a countable subset with empty complement.

Our notion of ordinal is that of the HoTT book

An ordinal is a type X with a transitive, extensional, accessible relation $(-) < (-) : X \to X \to U$.

1. Extensional means that any two elements with the same predecessors are equal.

2. The accessibility of points of X is inductively defined.

   We say that $x : X$ is accessible whenever every $y < x$ is accessible.

   The accessibility of a point is a subsingleton.

3. $<$ is accessible if every $x : X$ is accessible.

   The accessibility of $<$ implies that it is subsingleton valued, and that X is set.

Side remark

A functor $F : U \to U$

$F(X) = L(\lambda n.X)$, which is equivalent to $\Sigma(u : \mathbb{N}_\infty).\Pi(n : \mathbb{N}).X^{\underline{n}=u}$.

An equivalent coinductive definition of F is given by constructors

$$\begin{aligned} \text{zero} &: X \to F(X), \\ \text{succ} &: F(X) \to F(X). \end{aligned}$$

1. The Cantor type $2^{\mathbb{N}}$ is the carrier of a final coalgebra of F.
2. There is an initial algebra, whose carrier is the subset of Cantor consisting of the sequences with finitely many zeros, for a suitable notion of finiteness.

   (Which is classically equivalent to the classical one.)

For the sake of completeness, we characterize the injectives in UF

We have seen that universes are injective, and applied this to construct searchable types and ordinals.

Independently of this, it is natural to try to grasp the injective types.

1. In topos theory, the injectives are the retracts powers of the subobject classifier.

2. We show that, in UF, they are the retracts of powers of universes.

3. Before concluding, we prove this and offer a finer analysis.

The Yoneda embedding

1. For any type X, point $x : X$ and family $A : X \to U$,

   $$(\Pi(y : X).\, \mathrm{Id}\; x\, y \to A(y)) \simeq A(x).$$

   This is the Yoneda Lemma.

2. Say that A is representable if we have $x : X$ with $A(y) \simeq \mathrm{Id}\; x\, y$.

3. A having a universal element amounts to $\Sigma(x : X).A(x)$ being contractible.

4. The representability of A is equivalent to the contractibility of $\Sigma(x : X).A(x)$, and hence representability is a proposition.

Therefore, assuming univalence,

Theorem. *The map $\mathrm{Id} : X \to U^X$ is an embedding.*

Standard reasoning with injectives

1. Any power I^X of an injective type I is again injective.

   This argument needs that if $A \to B$ is an embedding then so is its product $A \times X \to B \times X$ with the identity map.

2. A retract of an injective type is again injective.

3. An injective type is a retract of every type in which it is embedded.

Characterization of the injective types

Combining this with the Yoneda Embedding:

Theorem. *The injective types are precisely the retracts of powers of the universes.*

We also have:

Theorem. *The injective sets are precisely the retracts of powers of the universe of propositions.*

Theorem. *The injective $n+1$-types are precisely the retracts of powers of the universe of n-types.*

To conclude, we want more precise models of constructive univalent type theories

1. The simple types $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ are discrete in the simplicial set model, but have a rich topology as Kleene–Kreisel spaces (and live in the topological topos as such).

2. This topology can be "internally detected" e.g. via a synthetic notion of compactness given by searchability, but also in many other "constructive" ways not discussed here.

3. If one wishes to simultaneosly understand

   3.1 constructivity

   (approximated by continuity),

   3.2 identity types

   (which give types the structure of ω-groupoids),

   then one needs a model which can simultaneously see e.g. $\mathbb{N}_\infty$ as compact (as a space) and discrete (as an ω-groupoid).

We want more precise models of constructive univalent type theories

1. We seem to need "topological ω-groupoids" (whatever they may be) to better understand constructivity in the presence of identity types.
2. E.g. The simplicial set model makes $\mathbb{N}_\infty$ discrete in the two senses and (hence) non-compact!
3. The topological models do a good job capturing constructivity, even if not precisely, but the simplicial set model is oblivious to constructivity.

- I hope the above development illustrates this.
- We looked at a synthetic notion of topological compactness, and relied on univalent concepts, ideas, and axioms to investigate it.

Reference for higher-type computation

John Longley and Dag Normann.
Higher-Order Computability. Springer, 2015.

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