Negative consistent axioms can be postulated without loss of canonicity

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Abstract

We show that, in intensional Martin-Löf type theory, negative consistent axioms can be postulated so that every closed term of natural number type reduces to a numeral. We also sketch some applications.

1 Introduction

Intensional Martin-Löf type theory (MLTT) is a compelling minimal foundation for mathematics, which is simultaneously a programming language, and hence constructive in a strong sense. A number of well-known practical proof assistants and/or programming languages are based on variations of MLTT.

Compared to classical (non-constructive) mathematics, it lacks excluded middle. Compared to Brouwer's intuitionism, it lacks continuity axioms and bar induction. Compared to topos theory, it lacks impredicativity and has a different treatment of the quantifiers. Compared to Homotopy Type Theory (HoTT), it lacks the univalence axiom and higher-inductive types. Moreover, a fundamental property of MLTT's propositional equality with respect to function types, namely function extensionality, which is desirable in all examples of mathematics discussed above (and which follows from univalence), is lacking in MLTT, as is well known.

All of these mathematical theories can be accomodated in MLTT by simply postulating the required "missing" axioms (impredicativity requires a more careful treatment, but the topos-theoretic quantifiers can be reduced to the MLTT quantifiers via (postulated) propositional reflections as in HoTT).

However, once an axiom is postulated, MLTT loses the fundamental *canonicity* property that every closed term of natural number type reduces to a numeral. Thus, in a practical manifestation of MLTT, after a (consistent) axiom is postulated, one still has a proof assistant, but the proof language ceases to be a programming language, which, as discussed above, is a major compelling aspect of MLTT compared to non-constructive foundations such as set theory.

There are several ways out of this obstacle, some of which have been successful in particular directions (e.g. how to realize classical countable (dependent) choice, via variations of bar recursion, or how to realize function extensionality, via setoids), and some of which are the subject of active research (e.g. how to realize univalence, how to realize higher-inductive types).

In this note we discuss a cheap way of postulating axioms in (intensional) MLTT so that canonicity is not lost. This technique has limited scope, but nevertheless is applicable to a number of interesting examples. We observe that if a collection of consistent axioms of the form $\neg A$ are postulated, then canonicity is preserved (Section 2), and we sketch examples where this is useful (Section 3), which will be fully developed elsewhere, by different subsets of the list of authors in this note.

2 Canonicity of negative, consistent postulates

By MLTT we mean any of the received variations of intensional Martin-Löf type theory that enjoy the strong normalization property. These variations may or may not include universes, in a couple of flavours, or several forms of generalized inductive definitions of types that have been put forward after Martin-Löf's established the bones of the theory.

We say that a collection A_n of closed syntactical types is *jointly consistent* with MLTT if the extension of MLTT with constants $c_n : A_n$ doesn't inhabit the empty type with a closed term. By a *numeral* we mean a closed term of natural number type $\mathbb{N}$ of the form $\mathrm{succ}^n(0)$. We will write, as usual, $\neg A$ as a shorthand for the type $A \to \emptyset$, where $\emptyset$ is the empty type.

Meta-Theorem 1 *If $\neg B_n$ is a finite or countably infinite collection of closed types jontly consistent with MLTT, then every closed term of type $\mathbb{N}$ in MLTT extended with constants $c_n : \neg B_n$ reduces to a numeral.*

To prove this, consider $A_n = \neg B_n$ in the definition of joint consistency. If the collection A_n is jointly consistent, then, for any n, there is no closed term $\phi_n : \neg\neg B_n$, for if there were, the closed term $\phi_n(c_n)$ would inhabit the empty type. Hence there is no closed term of type B_n either, because $B_n \to \neg\neg B_n$. One approach to the proof is to use the normalization theorem, which holds when any collection of constants is added, and to analyze the normal forms of closed terms of type $\mathbb{N}$ with constants $c_n : \neg B_n$. As there is no closed term of type B_n, any such normal form has to be a numeral. An alternative approach is to re-prove the normalization theorem from scratch, using any of the known generalizations of Tait's computability method for system T to MLTT, where the base case of the inductive definition of computability includes that any closed term of type $\mathbb{N}$ reduces to a numeral. Then one needs to extend the proof with the inductive clause that if a closed term $b_n : B_n$ is computable, then so is $c_n(b_n)$, but this is vacuously true, because there is no such closed term b_n, as we have seen.

Some remarks and questions are in order. (1) In the applications given below, the family consists of just one or two axioms. (2) The theorem for the unary case gives the theorem for the finite case, as the reader can easily verify, by packing finitely many axioms into a single one. (3) Moreover, as is also easy to see, the countably infinite case also follows from the unary case, provided the collection $\neg B_n$ is uniform, in the sense that there is $B : \mathbb{N} \to \mathrm{Type}$ such that $B_n = B\,(\mathrm{succ}^n(0))$. The force of our formulation of the theorem is that the variable n in the meta-theorem ranges over natural numbers in the meta-language, and there is no uniformity requirement, although at present we don't have any useful example to offer. (4) The theorem discusses only numerals. It would be interesting to see a more complete characterization of the types for which canonicity is not lost. Possibly canonicity holds whenever the outermost type constructor is something other than $\prod$. Closed types headed by $\emptyset$, 1, $+$, $\sum$, W and propositional equality should all work. Nevertheless, the given formulation of the theorem is enough to retain the programming-language character of the theory extended with consistent negative axioms.

3 Examples

Example 1. Danielsson (unpublished) considered the following two definitions of "true infinitely often", for a predicate $P : \mathbb{N} \to \text{Type}$:

$$\begin{array}{rcl} \text{Inf}_1 : (\mathbb{N} \to \text{Type}) & \to & \text{Type} \\ P & \mapsto & \neg(\exists i, \forall j, i \leq j \to \neg(Pj)), \end{array}$$

$$\begin{array}{rcl} \text{Inf}_2 : (\mathbb{N} \to \text{Type}) & \to & \text{Type} \\ P & \mapsto & \forall i, \exists j, i \leq j \wedge Pj. \end{array}$$

Inf_2 is "more constructive" (gives more information), and $\text{Inf}_2\, P$ implies $\text{Inf}_1\, P$. They proved that

$$\forall (P : \mathbb{N} \to \text{Type}) \to \text{Inf}_1\, P \to \neg\neg(\text{Inf}_2\, P)$$

is logically equivalent to the following double-negation shift property:

$$\forall (P : \mathbb{N} \to \text{Type}) \to (\forall i \to \neg\neg(Pi)) \to \neg\neg(\forall i \to Pi).$$

This property can be proved in a system with a suitable kind of bar recursion. Norell suggested that, for this application, it would be easier to simply postulate the double negation of excluded middle, and that a consistent postulate without computational content wouldn't break canonicity.

Example 2. Escardó and Xu considered a construction in MLTT of a sheaf model of type theory that validates the axiom that all functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous, where 2 is the two-point type of binary digits and $2^{\mathbb{N}} = (\mathbb{N} \to 2)$ is the Cantor type. This proof, developed in Agda and published in TLCA'2013, relies on the axiom of function extensionality (any two pointwise equal functions are equal). However, they noticed that the double negation of function extensionality is enough, and they independently posited that postulating a negative axiom wouldn't destroy canonicity, and Xu also implemented the proof in Agda with the double negation of function extensionality as a postulate after the TLCA publication.

Example 3. Escardó proved (Journal of Symbolic Logic, 2013) that the set $\mathbb{N}_\infty$ of decreasing binary sequences satisfies

$$\forall p : \mathbb{N}_\infty \to 2, (\exists x : \mathbb{N}_\infty, p(x) = 1) \vee (\forall x : \mathbb{N}_\infty, p(x) = 0).$$

Two proofs, when rendered in type theory (developed in Agda notation), either assume that p is extensional (it has the same value on pointwise equal arguments) or else that the axiom of function extensionality holds. However, the double negation of function extensionality suffices, and Xu confirmed this by modifying the Agda proof. Thus the above theorem has computational content when rendered in type theory.

Example 4. Escardó proved (to appear in Mathematical Structures in Computer Science) that any function $f : \mathbb{N}_\infty \to \mathbb{N}$ is either not continuous or not-not continuous. This again assumes function extensionality, when rendered in MLTT, but the double negation of function extensionality suffices (this was found after we knew of Examples 2 and 3). Moreover, non-continuity of any particular function $f : \mathbb{N}_\infty \to \mathbb{N}$ is equivalent to WLPO, and doubly negated continuity for all f is equivalent to $\neg$WLPO. Thus, if we further postulate $\neg$WLPO, all functions $f : \mathbb{N}_\infty \to \mathbb{N}$ are not-not continuous, and canonicity is not lost with this.

Notice that some of the examples are mutually inconsistent: the double negation of excluded middle and the negation of WLPO contradict each other.