INFINITE SETS THAT SATISFY THE PRINCIPLE OF OMNISCIENCE IN ALL VARIETIES OF CONSTRUCTIVE MATHEMATICS

MARTÍN H. ESCARDÓ

Abstract. We show that there are plenty of infinite sets that satisfy the omniscience principle, in a minimalistic setting for constructive mathematics that is compatible with classical mathematics. A first example of an omniscient set is the *one-point compactification of the natural numbers*, also known as the *generic convergent sequence*. We relate this to Grilliot's and Ishihara's Tricks. We generalize this example to many infinite subsets of the Cantor space. These subsets turn out to be ordinals in a constructive sense, with respect to the lexicographic order, satisfying both a well-foundedness condition with respect to decidable subsets, and transfinite induction restricted to decidable predicates. The use of simple types allows us to reach any ordinal below ϵ_0, and richer type systems allow us to get higher.

§1. Introduction. We show that there are plenty of infinite sets X that satisfy the omniscience principle

$$\text{for every function } p\colon X \to 2,\ \exists x \in X(p(x) = 0) \vee \forall x \in X(p(x) = 1).$$

For X finite this is trivial, and for $X = \mathbb{N}$, this is LPO, the limited principle of omniscience, which of course is and will remain a taboo in any variety of constructive mathematics [5, 2, 21]. A first example of an infinite omniscient set is the *one-point compactification* of $\mathbb{N}$,

$$\mathbb{N}_\infty = \{x \in 2^{\mathbb{N}} \mid \forall i \in \mathbb{N}(x_i \geq x_{i+1})\},$$

also known as the *generic convergent sequence*. We generalize this to many other subsets of the Cantor space $2^{\mathbb{N}}$. These subsets turn out to be ordinals in a constructive sense, with respect to the lexicographic order, satisfying both a well-foundedness condition with respect to decidable subsets, and transfinite induction restricted to decidable predicates. The use of simple types allows us to dominate any ordinal below ϵ_0, and richer type systems allow us to get higher, applying [8].

We work in a spartan constructive setting that is compatible with classical mathematics and *does not* postulate or reject contentious axioms such as choice, power set, Markov's principle, double-negation shift, continuity, bar induction, fan theorem, or Church's thesis. The development crucially relies on extensionality, but we do not postulate it as an axiom. We instead use it as an assumption, reserving the terminology *function* and *map* to refer to extensional operations, as in Bishop [3], where every set comes equipped with a given notion of equality. Similarly, subsets are taken to be extensional by definition. A formal illustration of this treatment of extensionality is given by a development of Section 3 in intensional Martin-Löf type theory, written in Agda notation [4], available at [11].

We need some amount of higher types, at least level 3 to develop the main construction (countable squashed sums) that allows us to build increasingly more complex omniscient sets of type 1. With the help of a further construction [8], the higher we climb the type hierarchy, the more complex omniscient sets of type 1 we get. A minimal formal system for our development is HA^ω, and all constructions developed here can be directly seen as programs in Gödel's system T.

Everything we say is of course a triviality from the point of view of classical mathematics, although perhaps unbelievable at first sight from the point of view of minimalistic constructive mathematics, despite the fact that similarly outrageous facts, with related constructions, are developed by e.g. Brouwer [22, page 459], Kreisel–Lacombe–Shoenfield [19], Bishop [3, page 177], Grilliot [15], Kreisel [1, page 581, Exercise 1], Bergstra [1, page 581, Exercise 2], Ishihara [17]. See the discussion in Section 12.

This paper is organized as follows: (2) selection of roots of 2-valued functions, (3) searchability of the generic convergent sequence, (4) fundamental properties of omniscient sets, (5) general facts about $\mathbb{N}_\infty$, (6) Ishihara's Tricks from the omniscience of $\mathbb{N}_\infty$, (7) Grilliot's Trick from a constructive perspective, (8) the nature of the maps $\mathbb{N}_\infty \to 2$ in the absence of continuity axioms, (9) ordinals in the Cantor space, (10) squashed sums of searchable sets and of ordinals in the Cantor space, (11) meta-mathematical discussion, (12) related work and acknowledgements.

§2. Selection of roots of 2-valued functions. The omniscience principle for a set X says that it is decidable whether a given map $p\colon X \to 2$ has a root. For some sets X, given any function $p\colon X \to 2$, it is possible to construct a putative root $x \in X$ such that p has a root if and only if x is indeed a root. The simplest, non-trivial, and perhaps surprising example of such a set is $X = \mathbb{N}_\infty$, as shown in Section 3. In this section we briefly investigate general aspects of this phenomenon and its relation to omniscience, to be exploited in the following sections.

DEFINITION 2.1 (Selection function, searchable set). A *selection function* for a set X is a functional $\varepsilon\colon (X \to 2) \to X$ such that for all maps $p\colon X \to 2$,

$$p(\varepsilon(p)) = 1 \Longrightarrow \forall x \in X (p(x) = 1).$$

We call a set *searchable* if it has a selection function.

LEMMA 2.2. *Any searchable set satisfies the omniscience principle.*

PROOF. This follows from the definition of selection function, using the facts that $p(\varepsilon(p)) = 0$ implies $\exists x \in X(p(x) = 0)$ by considering $x = \varepsilon(p)$, and that either $p(\varepsilon(p)) = 0$ or else $p(\varepsilon(p)) = 1$ by the decidability of equality of the set $2 = \{0, 1\}$ of binary numbers. ⊣

Moreover, assuming a selection function:

LEMMA 2.3. *A map p has a root if and only if $\varepsilon(p)$ is a root.*

PROOF. If p has a root x, that is, $p(x) = 0$, then $\neg\forall x \in X(p(x) = 1)$, and hence the contra-positive of the definition of selection function gives $p(\varepsilon(p)) \neq 1$, and therefore $p(\varepsilon(p)) = 0$, which shows that $\varepsilon(p)$ is a root. ⊣

This shows that if X is searchable, then there is a functional $E\colon (X \to 2) \to 2$ whose roots p are the functions that have roots,

$$E(p) = 0 \iff \exists x \in X(p(x) = 0),$$

constructed as

$$E(p) = p(\varepsilon(p)).$$

PROPOSITION 2.4. *If Y is searchable, the choice principle*

$$(\forall x \in X \exists y \in Y(A(x,y))) \Longrightarrow \exists f\colon X \to Y \forall x \in X(A(x, f(x))$$

holds for any set X and any decidable propositional function $A(x,y)$.

PROOF. Let $f(x) = \varepsilon_Y(y \mapsto a(x,y))$ where $a\colon X \times Y \to 2$ is the function whose roots are the pairs related by A, which exists by the decidability of $A(x,y)$. ⊣

The notion of selection function is investigated in [10, 14, 13] from various points of view, sometimes switching the roles of 0 and 1, but this is of course unimportant as it is simply a matter of naming conventions. Sets that satisfy the omniscience principle are called *exhaustible* in [10], where they are studied in the context of classical higher-type computability theory with partial continuous functionals. As discussed in the introduction, here our reasoning is purely constructive, under a minimalistic foundation for constructive mathematics, where all functions are tacitly total and not assumed to be continuous.

§3. Searchability of the generic convergent sequence.

We work with extensionally defined equality on the Cantor space $2^{\mathbb{N}}$ and hence on its subset $\mathbb{N}_\infty$,

$$\alpha = \beta \iff \forall i \in \mathbb{N}(\alpha_i = \beta_i),$$

and with the standard apartness defined by

$$\alpha \sharp \beta \iff \exists i \in \mathbb{N}(\alpha_i \neq \beta_i).$$

We will use the symbol $\neq$ for negation of equality, rather than apartness as often done in Bishop mathematics. The following shows that any map $\mathbb{N}_\infty \to X$ extends to a map $2^{\mathbb{N}} \to X$ with the same image:

PROPOSITION 3.1. *The set $\mathbb{N}_\infty$ is a retract of $2^{\mathbb{N}}$ in the sense that there is an idempotent map $r\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with image $\mathbb{N}_\infty$.*

PROOF. For example, we can let $r(\alpha)(n) = \min\{\alpha_k \mid k \leq n\}$. ⊣

The set $\mathbb{N}_\infty$ has elements

$$\underline{n} = 1^n 0^\omega, \qquad \infty = 1^\omega,$$

defined in more conventional form by $\underline{n}_i = 1 \iff i < n$ and $\infty_i = 1$. But it is a taboo to assert that $\mathbb{N}_\infty = \underline{\mathbb{N}} \cup \{\infty\}$, where of course $\underline{\mathbb{N}} = \{\underline{n} \mid n \in \mathbb{N}\}$. Recall the following definitions (limited principle of omniscience, weak limited principle of omniscience, and Markov's principle) where α ranges over $2^{\mathbb{N}}$:

$$\begin{aligned} \text{LPO} &\iff \exists n \in \mathbb{N}(\alpha_n = 0) \vee \forall n \in \mathbb{N}(\alpha_n = 1), \\ \text{WLPO} &\iff \forall n \in \mathbb{N}(\alpha_n = 1) \vee \neg\forall n \in \mathbb{N}(\alpha_n = 1), \\ \text{MP} &\iff \neg\forall n \in \mathbb{N}(\alpha_n = 1) \Longrightarrow \exists n \in \mathbb{N}(\alpha_n = 0). \end{aligned}$$

Clearly LPO $\iff$ WLPO $\wedge$ MP. The first two are regarded as taboos in all varieties of constructive mathematics, and the third is considered dubious in some but not all varieties [5]. Notice that a sequence $\alpha \in 2^{\mathbb{N}}$ satisfies one of the above conditions if and only if the sequence $r(\alpha) \in \mathbb{N}_\infty$ satisfies the same condition, where $r\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is the retraction defined in Proposition 3.1. Hence these taboos and dubious fact can be formulated in terms of the sets $\mathbb{N}_\infty$ and $\underline{\mathbb{N}}$ as follows:

$$\begin{aligned} \text{LPO} &\iff \mathbb{N}_\infty = \underline{\mathbb{N}} \cup \{\infty\} &&\iff \forall x \in \mathbb{N}_\infty(x \sharp \infty \vee x = \infty), \\ \text{WLPO} &\iff \mathbb{N}_\infty = \mathbb{N}_\infty \setminus \{\infty\} \cup \{\infty\} &&\iff \forall x \in \mathbb{N}_\infty(x \neq \infty \vee x = \infty), \\ \text{MP} &\iff \mathbb{N}_\infty \setminus \{\infty\} = \underline{\mathbb{N}}, &&\iff \forall x \in \mathbb{N}_\infty(x \neq \infty \Longrightarrow x \sharp \infty). \end{aligned}$$

Some of the equivalences displayed above implicitly rely on the following:

LEMMA 3.2. *For any $x \in \mathbb{N}_\infty$, if $x_n = 0$ then $x = \underline{k}$ for some $k \leq n$.*

PROOF. The unique such k can be constructed as $\sum_{i<n} x_i$. ⊣

We say that a subset F of a set X is *full* if its complement is empty. Because we are not assuming the axiom of extensionality, we take the complement to be

$$X \setminus F = \{x \in X \mid \forall x' \in F(x \neq x')\}.$$

The following shows that $\underline{\mathbb{N}} \cup \{\infty\}$ is a full subset of $\mathbb{N}_\infty$:

LEMMA 3.3. $\forall x \in \mathbb{N}_\infty(\forall k \in \mathbb{N}(x \neq \underline{k})) \Longrightarrow x = \infty$.

PROOF. Let $x \in \mathbb{N}_\infty$, assume that $\forall k \in \mathbb{N}(x \neq \underline{k})$, and let $n \in \mathbb{N}$. If $x_n = 0$, then $x = \underline{k}$ for some k by Lemma 3.2, which contradicts the hypothesis, and hence we must have $x_n = 1$. Because n is arbitrary, $x = \infty$. ⊣

A set is *separated* if $\neg(x \neq y) \Longrightarrow x = y$. Metrizable sets and hence discrete sets (sets with decidable equality) are examples. The subset of positive, zero and negative real numbers is full, and hence the following observation shows that two real valued functions g, h on the reals can be shown to be equal by simply checking each of the above three possibilities, by considering $f(x) = g(x) - h(x)$, despite the fact that the Trichotomy Law $x < 0 \vee x = 0 \vee x > 0$ is a taboo.

LEMMA 3.4 (Full subsets are dense). *If a function $f\colon X \to Y$ with values on a separated set Y is constant on a full subset F of X, then it is constant on X.*

PROOF. Assume that f has value y at F. If we had $f(x) \neq y$ for $x \in X$, the extensionality of f would give $x \neq x'$ for every $x' \in F$, which contradicts the fullness of F. Hence $f(x) = y$ by the separatedness of Y. ⊣

Notice that this logical notion of density does not refer to any kind of topological structure.

THEOREM 3.5. *The set* $\mathbb{N}_\infty \subseteq 2^{\mathbb{N}}$ *has a selection function given by*

$$\varepsilon(p)(n) = \min\{p(\underline{k}) \mid k \leq n\}.$$

PROOF. This is clearly well defined (notice that $\varepsilon(p) = r(n \mapsto p(\underline{n}))$, where r is the retraction defined in Proposition 3.1). As $\varepsilon(p)(n) = 0$ iff $\exists k \leq n(p(\underline{k}) = 0)$, we have that

$$\varepsilon(p) = \underline{n} \iff p(\underline{n}) = 0 \wedge \forall k < n(p(\underline{k}) = 1), \tag{1}$$

$$\varepsilon(p) = \infty \iff \forall n \in \mathbb{N}(p(\underline{n}) = 1). \tag{2}$$

To prove that ε is a selection function for $\mathbb{N}_\infty$, let $p\colon \mathbb{N}_\infty \to 2$ be a map and assume that $p(\varepsilon(p)) = 1$. Then $\varepsilon(p) \neq \underline{n}$ for any $n \in \mathbb{N}$, for if we had $\varepsilon(p) = \underline{n}$ we would have $p(\underline{n}) = 1$ by the assumption and the extensionality of p, which contradicts (1). Hence $\varepsilon(p) = \infty$ by Lemma 3.3. This implies both $p(\infty) = 1$, by the assumption and the extensionality of p, and $p(\underline{n}) = 1$ for every $n \in \mathbb{N}$, by (2). Therefore $p(x) = 1$ for any $x \in \mathbb{N}_\infty$ by Lemmas 3.3 and 3.4. ⊣

COROLLARY 3.6. *The set* $\mathbb{N}_\infty$ *satisfies the principle of omniscience.*

The set $\mathbb{N} \cup \{\infty\}$ is clearly in bijection with $\mathbb{N}$, but:

COROLLARY 3.7. *If* $\mathbb{N}_\infty$ *is in bijection with* $\mathbb{N}$*, then LPO holds.*

By *Kreisel's Tricks* we mean the ideas attributed to Kreisel in Exercise 1 of [1, page 581]). They sketch how to (1) perform search over $\mathbb{N}_\infty$, in Gödel's system T assuming classical logic to prove correctness, and (2) use this to effectively decide whether a function has a particular kind of discontinuity, namely $p(\underline{n}) = 0$ for all $n \in \mathbb{N}$, but $p(\infty) = 1$. This section is a constructive reworking of trick (1), where the crucial new ideas are the notion of fullness and the logical density lemma. Trick (2) is developed in more generality in Lemma 6.3 below, as a consequence of the constructive version of (1), in the form of Ishihara's Tricks [17, 6, 7].

§4. Fundamental properties of omniscient sets. Call a set *discrete* if it has decidable equality. The Cantor space $2^{\mathbb{N}}$ and the Baire space $\mathbb{N}^{\mathbb{N}}$ cannot be discrete unless LPO holds, but the function spaces $2^{\mathbb{N}_\infty}$ and $\mathbb{N}^{\mathbb{N}_\infty}$ are discrete by Theorem 4.1(2):

THEOREM 4.1 (General properties).

1. *If* X *is omniscient, then* $\neg\neg\exists x \in X(p(x) = 0) \Longrightarrow \exists x \in X(p(x) = 0)$ *for any map* $p\colon X \to 2$.
2. *If* X *is omniscient and* Y *is discrete, then the functions* $X \to Y$ *under extensional equality form a discrete set.*
3. *If* X *and* Y *are omniscient, then for any map* $q\colon X \times Y \to 2$,

$$\exists x \in X \forall y \in Y(q(x, y) = 1) \vee \forall x \in X \exists y \in Y(q(x, y) = 0).$$

4. *If* X *is an omniscient ordered set with binary maximum* $\max\colon X \times X \to X$*, then for any map* $p\colon X \to 2$,

$$\exists x \in X \forall y \geq x(p(y) = 1) \vee \forall x \in X \exists y \geq x(p(y) = 0).$$

PROOF. (1): One always has $\neg\neg\exists x \in X(p(x) = 0) \Longrightarrow \neg\forall x \in X(p(x) = 1)$, and the implication $\neg\forall x \in X(p(x) = 1) \Longrightarrow \exists x \in X(p(x) = 0)$ follows directly from the definition of omniscience of X.

(2): $f = g$ iff $\forall x \in X(f(x) = g(x))$, and hence the claim follows by the omniscience of X applied to the map $p\colon X \to 2$ defined by $p(x) = 0 \iff f(x) = g(x)$ using the discreteness of Y.

(3): By the omniscience of Y, there is a function $p\colon X \to 2$ such that $p(x) = 1$ if $\exists y \in Y(q(x, y) = 0)$, and $p(x) = 0$ if $\forall y \in Y(q(x, y) = 1)$, and hence the result follows by the omniscience of X applied to p.

(4): This is an instance of (3) with $q(x, y) = p(\max(x, y))$. ⊣

THEOREM 4.2 (Basic closure properties).

1. *If X and Y are omniscient, then so is $X \times Y$.*
2. *If X is omniscient then so is its image $f(X)$ for any map $f\colon X \to Y$.*
3. *If X is omniscient then so is the union $\bigcup_{x\in X} Y_x$ of any X-indexed family of omniscient sets $Y_x \subseteq Z$.*
4. *If X is omniscient then so is any decidable subset $A \subseteq X$.*

PROOF. (1): The argument is the same as that of Theorem 4.1(3), using the fact that $\exists z \in X \times Y(q(z) = 0)$ is equivalent to $\exists x \in X \exists y \in X(q(x, y) = 0)$, but considering the function $p(x) = 0$ iff $\exists y \in Y(q(x, y) = 0)$ instead.

(2): $\exists y \in f(X)(q(y) = 0)$ is equivalent to $\exists x \in X(p(x) = 0)$ where $p(x) = q(f(x))$.

(3): $\exists z \in \bigcup_{x\in X} Y_x(q(x, y) = 0)$ is equivalent to $\exists x \in X \exists y \in Y_x(q(x, y) = 0)$.

(4): Given $p\colon A \to 2$, extend it to X by mapping all $x \notin A$ to 1, and apply the omniscience of X. ⊣

Some closure properties of searchable sets from [10, 14] are recalled later.

§5. General facts about $\mathbb{N}_\infty$. We now pause to collect together various facts for use in the following sections, which can be consulted on demand. They are about arithmetic, order, apartness, strong extensionality, continuity, discontinuity, and extensions to $\mathbb{N}_\infty$ of functions defined on $\mathbb{N}$. We have placed this section at this point rather than immediately after the introduction in order to emphasize that the omniscience of $\mathbb{N}_\infty$ can be proved rather directly.

5.1. Linear order and apartness. A *linear order* [21] on a set X is a binary relation $<$ satisfying

$$x < y \wedge y < z \Longrightarrow x < z,\ \neg(x < y \vee y < x) \iff x = y,\ x < y \Longrightarrow x < z \vee z < y.$$

These conditions imply that $x \not< x$ and that the relation defined by

$$x \mathrel{\sharp} y \iff x < y \vee y < x$$

is a tight apartness relation (the *intrinsic apartness*). This means that

$$\neg(x \mathrel{\sharp} x), \qquad x \mathrel{\sharp} y \Longrightarrow y \mathrel{\sharp} x, \qquad x \mathrel{\sharp} y \Longrightarrow x \mathrel{\sharp} z \vee z \mathrel{\sharp} y, \quad \neg(x \mathrel{\sharp} y) \Longrightarrow x = y,$$

where the last condition is called *tightness*. The axioms for a linear order also imply that the relation defined by

$$x \le y \iff y \not< x$$

is a partial order (the *intrinsic partial order*), that is, it is reflexive, transitive, and antisymmetric. Moreover, it satisfies

$$x < y \le z \Longrightarrow x < z, \quad x \le y < z \Longrightarrow x < z.$$

Notice that the negatively defined partial order $\le$ is positively characterized as

$$x \le y \iff (x \sharp y \Longrightarrow x < y) \iff (y < x \Longrightarrow x < y).$$

5.2. The lexicographic order. We often work with the family of equivalence relations $=_n$ on $2^{\mathbb{N}}$ defined by

$$\alpha =_n \beta \iff \forall i < n(\alpha_i = \beta_i).$$

The *lexicographic order* of $2^{\mathbb{N}}$, defined by

$$\alpha < \beta \iff \exists n \in \mathbb{N}(\alpha =_n \beta \wedge \alpha_n < \beta_n),$$

is clearly linear. Notice that there is at most one such n, and that

$$\alpha < \beta \iff \exists s \in 2^*(s0 \sqsubseteq \alpha \wedge s1 \sqsubseteq \beta),$$

where 2^* is the set of finite sequences and $\sqsubseteq$ denotes the prefix relation.

5.3. Basic arithmetic and order on $\mathbb{N}_\infty$. For $x \in \mathbb{N}_\infty$, define $x + 1 \in \mathbb{N}_\infty$ by cases as

$$(x+1)_0 = 1, \qquad (x+1)_{n+1} = x_n.$$

Then $\underline{n} + 1 = \underline{n+1}$ and $\infty + 1 = \infty$, and also

$$x + 1 = x \Longrightarrow x = \infty, \text{ and } x + 1 = \infty \Longrightarrow x = \infty.$$

Notice that the lexicographic order on $\mathbb{N}_\infty$ agrees with the pointwise order,

$$x \le y \iff x_i \le y_i \text{ for all } i.$$

Hence the minimum and maximum functions $\min, \max \colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ are given pointwise,

$$\min(x,y)(i) = \min(x_i, y_i), \qquad \max(x,y)(i) = \max(x_i, y_i),$$

and $x \le y \iff \min(x,y) = x \iff \max(x,y) = y$. Notice that

$$\min(\underline{m}, \underline{n}) = \underline{\min(m,n)}, \qquad \min(\underline{m}, \infty) = \min(\infty, \underline{m}) = \underline{m},$$
$$\max(\underline{m}, \underline{n}) = \underline{\max(m,n)}, \qquad \max(\underline{m}, \infty) = \max(\infty, \underline{m}) = \infty.$$

We frequently work with the function

$$\min \colon \mathbb{N}_\infty \times \mathbb{N} \to \mathbb{N}$$

defined by

$$\min(x, n) = \sum_{i<n} x_i = \sup\{i \in [0, n] \mid x_i = 1\},$$

where of course the sup of the empty set is 0. We have the expected properties

$$\min(\underline{m}, n) = \min(m, n), \qquad \min(\infty, n) = n, \qquad \underline{\min(x,n)} = \min(x, \underline{n}).$$

For $x, y \in \mathbb{N}_\infty$, one has that $\underline{n} < x \iff x_n = 1$ and $x \le \underline{n} \iff x_n = 0$, and so

$$x < y \iff \exists n(x_n = 0 \wedge y_n = 1) \iff \exists n(x \le \underline{n} < y).$$

Also $\underline{m} < \underline{n}$ in $\mathbb{N}_\infty$ iff $m < n$ in $\mathbb{N}$, and $\underline{n} < \infty$ for every $n \in \mathbb{N}$.

5.4. Apartness and strong extensionality on $\mathbb{N}_\infty$. The intrinsic apartness of the lexicographic order coincides with the standard apartness of $2^{\mathbb{N}}$ and $\mathbb{N}_\infty$ defined in Section 3. We work with the negation-of-equality apartness on discrete sets. Notice that, for $x \in \mathbb{N}_\infty$,

$$x \sharp x+1 \Longrightarrow \exists n \in \mathbb{N}(x = \underline{n}).$$

In fact, by definition of the standard apartness relation, x and $x+1$ differ at some index, and so one of them (and hence the other) is apart from ∞. So it remains to see that

$$x \sharp \infty \Longrightarrow \exists n \in \mathbb{N}(x = \underline{n}).$$

The hypothesis simply means that $x_i = 0$ for some i, and hence the conclusion follows from Lemma 3.3. Of course, which is very important for the results developed below, one cannot derive the same conclusion from the weaker hypothesis that $x \neq \infty$. Notice also that $\underline{m} \sharp \underline{n} \iff m \neq n$, so that the embedding $(n \mapsto \underline{n})\colon \mathbb{N} \to \mathbb{N}_\infty$ is strongly extensional. Recall that a function $f\colon X \to Y$ of sets equipped with apartness relations is called *strongly extensional* if it reflects apartness, in the sense that $f(x) \sharp f(x') \Longrightarrow x \sharp x'$. It is easy to see that the function $\max\colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ is strongly extensional in each argument:

$$\max(x,y) \sharp \infty \Longrightarrow x \sharp \infty.$$

The following is related to the fact that the condition $x = \underline{n}$ is decidable for any $x \in \mathbb{N}_\infty$ and $n \in \mathbb{N}$, as it amounts to $1^n 0 \sqsubseteq x$:

LEMMA 5.1. *A function $p\colon \mathbb{N}_\infty \to 2$ is strongly extensional iff $p(x) \neq p(\infty)$ implies $x \sharp \infty$ for every $x \in \mathbb{N}_\infty$*

PROOF. ($\Leftarrow$) Assume that $p(x) \neq p(y)$. Then $p(x) \neq p(\infty)$ or $p(y) \neq p(\infty)$ by co-transitivity. Assume the first case without loss of generality. Then $x \sharp \infty$ by the hypothesis, and $x \sharp y$ or $y \sharp \infty$ again by co-transitivity. In the first case we are done, and hence assume the second. Then there are $m, n \in \mathbb{N}$ with $x = \underline{m}$ and $y = \underline{n}$. If we had $m = n$, the extensionality of p would give $p(x) = p(\underline{m}) = p(\underline{n}) = p(y)$, and hence we must have $m \neq n$, which shows that $x \sharp y$, as required. ⊣

5.5. Continuity and discontinuity. We say that a map $p\colon \mathbb{N}_\infty \to 2$ is *continuous* if

$$\exists n \in \mathbb{N} \forall m \geq n (p(m) = p(\infty)),$$

where we write m to mean $\underline{m}$. Of course there are continuous maps:

PROPOSITION 5.2. *If a sequence $\alpha\colon \mathbb{N} \to 2$ is eventually constant, then it extends to a strongly extensional, continuous map $p\colon \mathbb{N}_\infty \to 2$.*

PROOF. The hypothesis is that $\exists n \in \mathbb{N} \forall m > n(\alpha_m = \alpha_n)$. Let $p(x) = \alpha_{\min(x,n)}$, where $\min\colon \mathbb{N}_\infty \times \mathbb{N} \to \mathbb{N}$ is defined in Section 5.3. Then $p(m) = \alpha_m$ if $m < n$, and $p(m) = \alpha_n$ if $m \geq n$, so that $p(m) = \alpha_m$ for every $m \in \mathbb{N}$. Also $p(\infty) = \alpha_n$, which shows that p is continuous. If $p(x) \neq p(\infty)$, that is, $\alpha_{\min(x,n)} \neq \alpha_n$, then $\min(x,n) \neq n$ and so there is $i < n$ with $x_i = 0$, and hence $x \sharp \infty$. Therefore p is strongly extensional by Lemma 5.1. ⊣

We say that $p\colon \mathbb{N}_\infty \to 2$ is *discontinuous* if

$$\forall n \in \mathbb{N} \exists m \geq n (p(m) \neq p(\infty)).$$

This is a strengthening and positive formulation of the negation of continuity. However, Corollary 6.4 below shows that, in the strongly extensional case, the two notions agree, with an application of the omniscience of $\mathbb{N}_\infty$.

When $2^{\mathbb{N}}$ is equipped with its usual metric and 2 with the discrete metric, a map $p\colon 2^{\mathbb{N}} \to 2$ is

1. *pointwise continuous* iff $\forall \alpha \in 2^{\mathbb{N}} \exists n \in \mathbb{N} \forall \beta \in 2^{\mathbb{N}} (\alpha =_n \beta \Longrightarrow p(\alpha) = p(\beta))$,
2. *uniformly continuous* iff $\exists n \in \mathbb{N} \forall \alpha, \beta \in 2^{\mathbb{N}} (\alpha =_n \beta \Longrightarrow p(\alpha) = p(\beta))$.

Of course, uniform continuity implies pointwise continuity, but, as is well known, the converse cannot be proved without Brouwerian axioms. However:

PROPOSITION 5.3. *The notions of continuity, pointwise continuity, and uniform continuity agree for maps $p\colon \mathbb{N}_\infty \to 2$.*

PROOF. First observe that for $x, y \in \mathbb{N}_\infty$, the condition $x =_n y$ is equivalent to $\min(x, n) = \min(y, n)$ where $\min\colon \mathbb{N}_\infty \times \mathbb{N} \to \mathbb{N}$ is defined in Section 5.3.

Pointwise continuity $\Longrightarrow$ continuity. Because p is continuous at ∞, there is $n \in \mathbb{N}$ such that $p(\infty) = p(x)$ whenever $\min(\infty, n) = \min(x, n)$, that is, whenever $x \geq n$, which amounts to the continuity of p.

Continuity $\Longrightarrow$ uniform continuity. We show that any continuity witness n is a modulus of uniform continuity. Assume $\min(x, n) = \min(y, n)$ for $x, y \in \mathbb{N}_\infty$. If $\min(x, n) < n$ then $x = \underline{\min(x, n)} = \underline{\min(y, n)} = y$ and so $p(x) = p(y)$ by extensionality. Otherwise $\min(x, n) = n = \min(y, n)$, and so $x, y \geq n$ and hence $p(x) = p(\infty)$ and $p(y) = p(\infty)$ by continuity, and again $p(x) = p(y)$, as required. ⊣

5.6. Extension of functions $\mathbb{N} \to \mathbb{N}$ to $\mathbb{N}_\infty \to \mathbb{N}_\infty$. The following is formulated, for the sake of simplicity, with a hypothesis stronger than needed to get the same conclusion. The conditions on g in the lemma are $m \leq n \Longrightarrow g(m) \leq g(n)$ and $n \leq g(n)$ respectively.

LEMMA 5.4. *Any monotone increasing, inflationary map $g\colon \mathbb{N} \to \mathbb{N}$ extends to a strongly extensional map $G\colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ with $G(\infty) = \infty$.*

PROOF. Let $G(x)(n) = 1 \iff n < g(\min(x, n+1))$, where the function $\min\colon \mathbb{N}_\infty \times \mathbb{N} \to \mathbb{N}$ is defined in Section 5.3. Then $G(x) \in \mathbb{N}_\infty$, because this amounts to $n + 1 < g(\min(x, n+2)) \Longrightarrow n < g(\min(x, n+1))$. We have that $G(\infty) = \infty$ because $G(\infty)(n) = 1$ for any n. The map G extends g, i.e. $G(\underline{k}) = \underline{g(k)}$, if and only if $G(\underline{k})(n) = 1 \iff \underline{g(k)}(n) = 1$, that is, $n < g(\min(k, n+1)) \iff n < g(k)$, which is seen as follows. ($\Rightarrow$): By monotonicity, $g(\min(k, n+1)) \leq g(k)$, and hence transitivity gives $n < g(k)$ from the hypothesis. ($\Leftarrow$): By inflationarity, $n < g(n+1)$ and hence the hypothesis gives $n < \min(g(k), g(n+1)) = g(\min(k, n+1)$, because g is monotone. Finally, to prove strong extensionality, assume $G(x) \sharp G(y)$, that is, $G(x)(n) \neq G(y)(n)$ for some n. Without loss of generality, assume that $G(x)(n) = 0$ and $G(y)(n) = 1$, that is, $g(\min(x, n+1)) \leq n < g(\min(y, n+1)$. Then $x \leq n < y$, and hence $x_n = 0$ and $y_n = 1$, which shows that $x \sharp y$, as required. ⊣

5.7. The set $\mathbb{N}_\infty$ is the generic convergent sequence. The extension constructed in the following lemma is continuous, and any strongly extensional continuous function $x\colon \mathbb{N}_\infty \to X$ defines a convergent sequence, but we do not need these two additional facts.

LEMMA 5.5. *Any Cauchy sequence $x\colon \mathbb{N} \to X$ in a complete metric space X extends to a strongly extensional map $x\colon \mathbb{N}_\infty \to X$ with $x_\infty = \lim_n x_n$.*

PROOF. For any $\alpha \in \mathbb{N}_\infty$ define a sequence $y = y^\alpha \in X^{\mathbb{N}}$ by induction as $y_0 = x_0$, $y_{n+1} = x_{n+1}$ if $\alpha_n = 1$ and $y_{n+1} = y_n$ if $\alpha_n = 0$. This is a Cauchy sequence, and by the completeness of X we can define $x_\alpha = \lim_n y_n^\alpha$, and the stated requirements are easily verified. ⊣

5.8. Choice and decidability. The following is folklore — see e.g. [13]:

LEMMA 5.6 (Using choice). *For any given $P, Q \subseteq X$ with $P \cup Q = X$, there are disjoint sets $P' \subseteq P$ and $Q' \subseteq Q$ with $P' \cup Q' = X$.*

PROOF. By the hypothesis, $\forall x \in X \exists y \in 2(y = 0 \Longrightarrow x \in P \wedge y = 1 \Longrightarrow x \in Q)$, and by choice, $\exists p\colon X \to 2(\forall x \in X(p(x) = 0 \Longrightarrow x \in P \wedge p(x) = 1 \Longrightarrow x \in Q)$. To conclude, let $P' = p^{-1}(0)$ and $Q' = p^{-1}(1)$. ⊣

§6. Ishihara's Tricks from the omniscience of $\mathbb{N}_\infty$. We claim that the following lemma is the essence of *Ishihara's First Trick* [6].

LEMMA 6.1. *If a map $p\colon \mathbb{N}_\infty \to 2$ is strongly extensional, then*

$$\exists n \in \mathbb{N}(p(n) \neq p(\infty)) \vee \forall n \in \mathbb{N}(p(n) = p(\infty)).$$

PROOF. Because the condition $p(x) = p(\infty)$ is decidable for any $x \in X$, the omniscience of $\mathbb{N}_\infty$ gives $\exists x \in \mathbb{N}_\infty(p(x) \neq p(\infty))$ or $\forall x \in \mathbb{N}_\infty(p(x) = p(x_\infty))$. If the first case holds, then $x \sharp \infty$ by the strong extensionality of p, and hence $x = n$ for some $n \in \mathbb{N}$, and so $p(n) \neq p(\infty)$ by the extensionality of p. If the second case holds, then in particular $\forall n \in \mathbb{N}(p(n) = p(\infty))$ by considering $x = n$. ⊣

The following corollary is *Ishihara's First Trick* [6, Proposition 1], with the following modifications:

1. We assume a disjointness condition.
2. We do not assume that X is a complete metric space.

The disjointness assumption is more restrictive, but allows us to avoid the axiom of choice, which is tacitly applied in [6]. We confine the application of choice to Lemma 5.6. On the other hand, without the metric assumption, and replacing convergent sequences by strongly extensional maps $x\colon \mathbb{N}_\infty \to X$, we are more general, in view of Lemma 5.5. A set is called *tight* if it has a tight apartness.

COROLLARY 6.2. *If P, Q are disjoint subsets of a tight set X with $P \cup Q = X$ and $x\colon \mathbb{N}_\infty \to X$ is a strongly extensional map with*

$$\forall y \in X(y \sharp x_\infty \vee y \notin Q),$$

then

$$\forall n \in \mathbb{N}(x_n \in P) \vee \exists n \in \mathbb{N}(x_n \in Q).$$

PROOF. Define $q\colon X \to 2$ by $q(x) = 0 \iff x \in Q$. Then $q(x_\infty) = 1$ and so the hypothesis amounts to $y \sharp x_\infty \vee q(y) = q(x_\infty)$, which is equivalent to the implication $q(y) \neq q(x_\infty) \Longrightarrow y \sharp x_\infty$. Hence the map $p = q \circ x$ is strongly extensional by Lemma 5.1, and the result follows from Lemma 6.1. ⊣

Assuming choice as in [6], we recover [6, Proposition 1], using projective covers if necessary, as in [20], and Lemmas 5.5 and 5.6. The following is derived from two nested applications of Lemma 6.1, using the idea of proof of Theorem 4.1.

LEMMA 6.3. *If $p\colon \mathbb{N}_\infty \to 2$ is strongly extensional, then*

$$\exists n \in \mathbb{N} \forall m \geq n (p(m) = p(\infty)) \vee \forall n \in \mathbb{N} \exists m \geq n (p(m) \neq p(\infty)).$$

That is, p is either continuous or discontinuous.

COROLLARY 6.4. *If a strongly extensional map $p\colon \mathbb{N}_\infty \to 2$ fails to be continuous, then it is discontinuous in the positive sense defined in Section 5.5.*

PROOF OF LEMMA 6.3. Define $q\colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to 2$ by $q(x, y) = p(\max(x, y))$. Because p and $\max\colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ are strongly extensional so is q. By Lemma 6.1 applied to the function $(y \mapsto q(x, y))\colon \mathbb{N}_\infty \to 2$, there is $r\colon \mathbb{N}_\infty \to 2$ such that

$$r(x) = 1 \iff \exists m \in \mathbb{N}(q(x, m) \neq q(x, \infty)),$$
$$r(x) = 0 \iff \forall m \in \mathbb{N}(q(x, m) = q(x, \infty)).$$

Then $r(\infty) = 0$ because otherwise $\exists m \in \mathbb{N}(q(\infty, m) \neq q(\infty, \infty))$, which would amount to $p(\infty) \neq p(\infty)$. Now assume that $r(x) \neq r(\infty)$, that is, $r(x) = 1$. Then $q(x, m) \neq q(x, \infty)$ for some $m \in \mathbb{N}$, which amounts to $p(\max(x, m)) \neq p(\infty)$, and the strong extensionality of p gives $\max(x, m) \sharp \infty$ and so $x \sharp \infty$. Hence r is strongly extensional by Lemma 5.1. By Lemma 6.1 applied to r, and expanding the definitions of q and r, using the fact that $r(\infty) = 0$, we conclude that

$$\exists n \in \mathbb{N} \forall m \in \mathbb{N}(p(\max(n, m)) = p(\max(n, \infty)))$$
$$\vee \forall n \in \mathbb{N} \exists m \in \mathbb{N}(p(\max(n, m)) \neq p(\max(n, \infty))),$$

which is equivalent to the desired conclusion. ⊣

Lemma 6.3 amounts to a form of *Ishihara's Second Trick* [6, Proposition 2], with the same modifications discussed in the paragraph preceding Theorem 6.2, and, moreover, with a weaker hypothesis than that of [6, Proposition 2]:

COROLLARY 6.5. *If P, Q are disjoint subsets of a tight set X with $P \cup Q = X$ and $x\colon \mathbb{N}_\infty \to X$ is a strongly extensional map with*

$$\forall y \in X(y \sharp x_\infty \vee y \notin Q),$$

then

$$\forall n \in \mathbb{N} \exists m \geq n(x_m \in P) \vee \exists n \in \mathbb{N} \forall m \geq n(x_m \in Q).$$

PROOF. Literally the same as that of Theorem 6.2, but using Lemma 6.3 rather than 6.1 in the final step. ⊣

We again use Lemmas 5.5 and 5.6 to get [6, Proposition 2] as a corollary.

§7. Grilliot's Trick from a constructive perspective. In the context of higher-type recursion theory developed within classical mathematics, Grilliot [15, Lemma 1] showed that one can effectively define a functional $E : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ such that

$$E(h) = 0 \iff \exists n \in \mathbb{N}(h(n) = 0)$$

from any effectively discontinuous $F \colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$. This is known as *Grilliot's Trick* in the research community, and sometimes in print, as in e.g. [16]. Here F is called effectively discontinuous if there is a sequence $g_i \colon \mathbb{N} \to \mathbb{N}$ with limit f, with both g_i and f recursive in F, such that $F(f) \neq \lim_i F(g_i)$.

We reproduce Grilliot's argument, for comparison with a constructive counterpart given below. By taking a subsequence, we may assume that $F(f) \neq F(g_i)$ for every i. Again by taking a subsequence, we may assume that $g_i(j) = f(j)$ for all $j \leq i$. Notice that, in both cases, one needs unbounded search to find the next element of the subsequence. If one now defines $J \colon (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$ by

$$J(h)(j) = g_i(j), \text{where } i = \inf\{n \in [0, j] \mid h(n) = 0\},$$

and where the infimum of the empty set is of course the largest element j of the integer interval $[0, j]$, then

$$J(h) = \begin{cases} f & \text{if } \forall n \in \mathbb{N}(h(n) \neq 0), \\ g_i & \text{if } h(i) = 0 \text{ and } \forall n < i(h(n) \neq 0). \end{cases}$$

Therefore E can be defined by

$$E(h) = 0 \iff F(J(h)) \neq F(f).$$

Although the definitions are explicit, their correcteness proofs rely on classical logic (Hartley [16] addresses this to some extent).

We offer Theorem 7.1 below as a constructive counter-part of Grilliot's Trick, where we take discontinuity, as defined in Section 5.5 and investigated in Section 6, as a constructive replacement of the notion of effective discontinuity. With this stronger (classically equivalent) notion, we avoid unbounded search, but we have not managed to get away without countable choice. Another difference is our consideration of maps $p \colon \mathbb{N}_\infty \to 2$ rather than $F \colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$, but this is inessential.

THEOREM 7.1. *Assuming countable choice:*

1. *If there is a discontinuous map* $p \colon \mathbb{N}_\infty \to 2$*, then WLPO holds.*
2. *If there is a discontinuous strongly extensional map* $p \colon \mathbb{N}_\infty \to 2$*, LPO holds.*

PROOF. An application of choice to the hypothesis gives a modulus of discontinuity $g \colon \mathbb{N} \to \mathbb{N}$ with $g(n) \geq n$ and $p(g(n)) \neq p(\infty)$. We can ensure, with search bounded by $g(n)$ after choice is applied, that $g(n)$ is the least $m \geq n$ with $p(m) \neq p(\infty)$, so that g can be extended to a strongly extensional map $G \colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ with $G(\infty) = \infty$ by Lemma 5.4. To prove WLPO we show that $x = \infty$ or $x \neq \infty$ for any $x \in \mathbb{N}_\infty$, and to prove LPO we show that $x = \infty$ or $x \sharp \infty$. We reduce these tests to the decidable condition $p(G(x)) = p(\infty)$:

(*i*) $p(G(x)) = p(\infty)$: Then $p(G(x)) \neq p(G(n))$ for any $n \in \mathbb{N}$ because $p(G(n)) = p(g(n)) \neq p(\infty)$, and hence $x \neq n$ by the extensionality of the map $p \circ G$. Therefore $x = \infty$ by Lemma 3.3.

(*ii*) $p(G(x)) \neq p(\infty) = p(G(\infty))$: Then $x \neq \infty$ by the extensionality of $p \circ G$, and $x \sharp \infty$ if p, and hence $p \circ G$, is strongly extensional. ⊣

Notice that this argument, due to Grilliot, *does not* apply the omniscience of $\mathbb{N}_\infty$, and *does not* have it as a corollary. The argument shows that the hypothetical existence of a discontinuous function entails a taboo, LPO, the omniscience of $\mathbb{N}$, rather than that of $\mathbb{N}_\infty$, under the assumption of strong extensionality, or just WLPO without the assumption. The omniscience of $\mathbb{N}_\infty$, which is a fact rather than a taboo, is proved with an essentially different argument. However, using the omniscience of $\mathbb{N}_\infty$ as in Corollary 6.4 to get discontinuity from non-continuity, we have (relying on countable choice):

COROLLARY 7.2. *If some strongly extensional map* $p\colon \mathbb{N}_\infty \to 2$ *fails to be continuous, then LPO must hold.*

§8. The nature of the maps $\mathbb{N}_\infty \to 2$ in the absence of continuity axioms. We now briefly investigate the collection of sequences $\alpha\colon \mathbb{N} \to 2$ that arise as restrictions of functions $p\colon \mathbb{N}_\infty \to 2$. Equivalently, we investigate the sequences $\alpha \in 2^{\mathbb{N}}$ that can be extended to functions $p \in 2^{\mathbb{N}_\infty}$. It turns out that, even in the absence of continuity axioms, they are fairly restricted in character. Eventually constant sequences constructively satisfy the LPO condition, and hence the WLPO condition too. Although without continuity axioms one cannot prove that only the eventually constant sequences $\alpha \in 2^{\mathbb{N}}$ can be extended to maps $p \in 2^{\mathbb{N}_\infty}$, we can show that only those that satisfy the WLPO condition can be extended (Theorem 8.2). We begin with a lemma that mixes quantification over $\mathbb{N}_\infty$ and $\mathbb{N}$.

LEMMA 8.1. *For any map* $p\colon \mathbb{N}_\infty \to 2$,

$$\exists x \in \mathbb{N}_\infty (x \neq \infty \land p(x) = 0) \lor \forall n \in \mathbb{N}(p(n) = 1).$$

PROOF. Because the condition $p(y) = p(y + 1)$ is decidable, the omniscience of $\mathbb{N}_\infty$ tells us that one of the following two cases holds:

$$(1) \quad \exists y \in \mathbb{N}_\infty (p(y) \neq p(y + 1)) \quad \lor \quad (2) \quad \forall y \in \mathbb{N}_\infty (p(y) = p(y + 1)).$$

(1) Then $y \neq \infty$, for if we had $y = \infty$, we would also have $y = y+1$ and hence $p(y) = p(y + 1)$ by extensionality. Hence we also have $y + 1 \neq \infty$, for if we had $y + 1 = \infty$ we would have $y = \infty$. Because one of $p(y)$ and $p(y + 1)$ must be zero, we conclude that $\exists x \in \mathbb{N}_\infty (x \neq \infty \land p(x) = 0)$.

(2) If $p(0) = 0$, then the example $x = 0$ shows that $\exists x \in \mathbb{N}_\infty (x \neq \infty \land p(x) = 0)$. Otherwise we conclude that $\forall n \in \mathbb{N}(p(n) = 1)$ by induction on n. ⊣

We now quantify over $\mathbb{N}$ only:

THEOREM 8.2. *For any map* $p\colon \mathbb{N}_\infty \to 2$,

$$\forall n \in \mathbb{N}(p(n) = 1) \lor \neg \forall n \in \mathbb{N}(p(n) = 1).$$

PROOF. If the second disjunct of the lemma holds, there is nothing to prove, and hence assume the first. By Lemma 3.3, the condition $x \neq \infty$ implies that $\neg\forall n \in \mathbb{N}(x \neq n)$. Therefore we conclude that $\neg\forall n \in \mathbb{N}(p(n) = 1)$ holds, for if we had $\forall n \in \mathbb{N}(p(n) = 1)$ we would have $\forall n \in \mathbb{N}(x \neq n)$ by extensionality and the fact that $p(x) = 0$, which would be a contradiction. ⊣

If a sequence $\alpha \in 2^{\mathbb{N}}$ can be extended to a strongly extensional map $p \in 2^{\mathbb{N}}$, then it must satisfy the LPO condition:

THEOREM 8.3. *For any strongly extensional map* $p\colon \mathbb{N}_\infty \to 2$,

$$\exists n \in \mathbb{N}(p(n) = 0) \vee \forall n \in \mathbb{N}(p(n) = 1).$$

PROOF. We again consider the cases (1) and (2) of Lemma 8.1.

(1) By the strong extensionality of p, we have that $y \sharp y + 1$ and hence $y = n$ for some $n \in \mathbb{N}$ by Section 5.3. Because one of $p(y)$ and $p(y + 1)$ must be zero, we conclude that $\exists n \in \mathbb{N}(p(n) = 0)$.

(2) The argument is literally the same as that of case (2) of Lemma 8.1. ⊣

Notice that this is similar to, but not quite the same as, Lemma 6.1. With the same kind of argument as in Lemma 6.3, using two nested applications of Theorem 8.3, we conclude that:

COROLLARY 8.4. *For any strongly extensional map* $p\colon \mathbb{N}_\infty \to 2$,

$$\exists n \in \mathbb{N}\forall m \geq n(p(m) = 1) \vee \forall n \in \mathbb{N}\exists m \geq n(p(m) = 0).$$

§9. Ordinals.

For the purposes of this investigation, we define the notion of *ordinal* by considering suitable formulations of well-ordering and transfinite induction that involve *decidability* conditions. This notion of ordinal classically coincides with the classical one, but, from the point of view of constructive mathematics, it is probably at the same time audacious and restrictive, and hence not suitable as a substitute of more standard constructive notions of (countable) ordinal encodings. Nevertheless, in this and the following section we will show that $\mathbb{N}_\infty$ and plenty of other subsets of the Cantor space are (omniscient) ordinals in our sense, under the lexicographic ordering.

Regarding the following definition, we are not able to show that a well-ordered set automatically satisfies the principle of transfinite induction, but we will not be surprised if the conclusion is found to hold under additional, classically vacuous, assumptions. Hence we live with a pinch of salt:

DEFINITION 9.1 (Ordinal). We take an *ordinal* to be a linearly ordered set that is well-ordered with respect to *decidable* subsets, in the sense that every inhabited, decidable subset has a least element, and satisfies the usual principle of transfinite induction, restricted to *decidable* predicates.

We do not assume that our constructive setting allows us to quantify over the subsets of a set or over predicates, but this definition is not problematic because the two conditions amount to

$$\begin{aligned}
&\forall \text{ maps } p\colon X \to 2,\\
&(\exists y \in X(p(y) = 0)) \Longrightarrow \exists x \in X(p(x) = 0 \wedge \forall y \in X(p(y) = 0 \Longrightarrow x \leq y)),\\
&(\forall x \in X(\forall y < x(p(y) = 0)) \Longrightarrow p(x) = 0) \Longrightarrow \forall x \in X(p(x) = 0)),
\end{aligned}$$

and we do tacitly assume that we can quantify over functions. By Lemma 3.4:

LEMMA 9.2. *If a linearly ordered set X has a full subset that satisfies transfinite induction for arbitrary predicates, then X satisfies transfinite induction for decidable predicates.*

To build finite ordinals in the lexicographic order of the Cantor space, we start from $0 = \emptyset$ and $1 = \{\underline{0}\}$, and for $X, Y \subseteq 2^{\mathbb{N}}$ we define $X + Y = 0X \cup 1Y$ by prefixing 0 and 1 to the elements of X and Y. The ordinal 0 is anomalous from the point of view of this investigation, because it is not searchable. But 1 is (with a unique selection function), and if X and Y are searchable then so is $X + Y$, and the construction can be performed so that if the selection functions for X and Y calculate infima of sets of roots, then so does that for $X + Y$, and hence we get all the finite ordinals embedded into $2^{\mathbb{N}}$.

We remark that we will not be able to embed into the Cantor space an ordinal that classically is ω (for the reasons discussed in Section 11). In fact, with classical eyes, we will be able to account for successor ordinals only, or equivalently ordinal intervals of the form $[0, \gamma]$. From this point of view, the situation for 0 is no longer anomalous, as this naturally excludes the empty set, even when $\gamma = 0$. Moreover, still classically, we will be able to account for countable ordinals only. The reason is that only countable ordinals can be embedded in the natural order of the real line, as is well known, and the Cantor space (continuously) order-embeds into the real line via Cantor's third-middle construction. Of course, $\mathbb{N}_\infty$ classically is $\omega + 1$, or equivalently $[0, \omega]$. We work with the selection function $\varepsilon\colon (\mathbb{N}_\infty \to 2) \to \mathbb{N}_\infty$ constructed in Theorem 3.5.

LEMMA 9.3. *For any map $p\colon \mathbb{N}_\infty \to 2$, the value $\varepsilon(p) \in \mathbb{N}_\infty$ is the infimum of the set of roots of p.*

PROOF. For any lower bound y of the set of roots, $y \le \varepsilon(p)$: If $p(\varepsilon(p)) = 0$ then $\varepsilon(p)$ is in the set and we are done. Otherwise the claim is vacuous because the set is empty by definition of selection function, and so we are done with the first part of the proof. To conclude, we show that $\varepsilon(p)$ is a lower bound of the set of roots. In order to show that $\varepsilon(p) \le x$ for any given root x, assume that $x < \varepsilon(p)$. By the definitions of ε and the order, there is $n \in \mathbb{N}$ such that (i) $x_n = 0$ and (ii) $\min\{p(\underline{k}) \mid k \le n\} = 1$. By (i) and Lemma 3.2, there is $j \le n$ with $x = \underline{j}$. By (ii) we have $p(\underline{k}) = 1$ for every $k \le n$, and hence $p(x) = p(\underline{j}) = 1$ by the extensionality of p, which contradicts the fact that x is a root. Discharging the assumption $x < \varepsilon(p)$, we conclude that $\varepsilon(p) \le x$, by definition of $\le$. ⊣

THEOREM 9.4. *$\mathbb{N}_\infty$ is an ordinal in the sense of Definition 9.1.*

PROOF. The well-orderedness condition follows from Lemma 9.3, because, taking Lemma 2.3 into account, it says that any map $p\colon \mathbb{N}_\infty \to 2$ has a least root if it has a root. The transfinite induction principle for p follows by Lemmas 3.3 and 9.2, because the full subset $\underline{\mathbb{N}} \cup \{\infty\}$ satisfies transfinite induction for arbitrary predicates, by induction on $\mathbb{N}$ and case analysis. ⊣

§10. Squashed sums. Given countably many searchable sets $X_n \subseteq 2^{\mathbb{N}}$, we show that their *squashed sum* that arises as their disjoint union with an added point at infinity is also searchable. We construct the squashed sum as a subset of $2^{\mathbb{N}}$, so that we can (transfinitely) iterate this procedure. We rescale and translate each set X_n by prefixing the finite sequence 1^n0 to its members, in order to make the sets disjoint, and at the same time smaller in diameter and arbitrarily close to the sequence $\infty = 1^\omega$ as n increases. The squashed sum can be described as the closure of $\bigcup_n 1^n 0 X_n$ in the Cantor space. With classical eyes, this is $\bigcup_n 1^n 0 X_n \cup \{\infty\}$, and if the sets X_n are ordinals then the squashed sum will be the successor of their ordinal sum. Constructively, it will be the case that the squashed sum of countably many ordinals is an ordinal, for ordinals in the sense of Section 9. The set $\mathbb{N}_\infty$ will be the squashed sum of the constant sequence $X_n = \{\underline{0}\}$, so that the results of this section will subsume the main theorems we formulated and proved for $\mathbb{N}_\infty$ in the previous sections (but see the discussion at the end of this section).

LEMMA 10.1. $\forall \alpha \in 2^{\mathbb{N}}(\forall n \in \mathbb{N}(1^n 0 \not\sqsubseteq \alpha) \Longrightarrow \alpha = \infty)$.

PROOF. For any k we have $\alpha_k = 1$, for if we had $\alpha_k = 0$ then we would have $1^n0 \sqsubseteq \alpha$ for some $n \le k$, which contradicts the hypothesis. ⊣

For countably many given sets $X_n \subseteq 2^{\mathbb{N}}$, we define their squashed sum by

$$\overline{\sum_n} X_n = \{\alpha \in 2^{\mathbb{N}} \mid \forall n \in \mathbb{N}(1^n 0 \sqsubseteq \alpha \Longrightarrow \alpha \in 1^n 0 X_n)\}.$$

Then $\infty \in \overline{\sum}_n X_n$ vacuously, and by construction $1^m 0 X_m \subseteq \overline{\sum}_n X_n$. Moreover:

LEMMA 10.2. $\forall \alpha \in \overline{\sum}_n X_n (\forall n \in \mathbb{N}(\alpha \notin 1^n 0 X_n) \Longrightarrow \alpha = \infty)$.

PROOF. For any n, if we had $1^n 0 \sqsubseteq \alpha$, then we would have $\alpha \in 1^n 0 X_n$ by definition of $\overline{\sum}_n X_n$, which contradicts the hypothesis. Hence the result follows from Lemma 10.1. ⊣

The n-tail of a sequence $\alpha \in 2^{\mathbb{N}}$ is $\alpha^{(n)} \in 2^{\mathbb{N}}$ defined by $\alpha^{(n)}_i = \alpha_{n+i}$.

PROPOSITION 10.3. *The retracts of the Cantor space are closed under the formation of squashed sums.*

PROOF. Given idempotent maps $r_n \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with image X_n, define $r \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $r(\alpha)(n) = 0 \iff \exists k \le n(1^k 0 \sqsubseteq \alpha \wedge (1^k 0 r_k(\alpha^{(k+1)}))_n = 0)$. ⊣

To know that $1^n 0 X_n$ is searchable if X_n is, we use the following construction applied to the rescaling-translation map $\alpha \mapsto 1^n 0 \alpha$. This construction appears in [10, Proposition 4.3] and is studied in more generality in [14], which also shows it to be the functor of a certain selection monad J with $JX = ((X \to 2) \to X)$, where the resulting selection function $\varepsilon_{f(X)}$ amounts to $Jf(\varepsilon_X)$.

LEMMA 10.4. *If X is searchable, then so is its image $f(X)$ for any function $f \colon X \to Y$.*

PROOF. Given a selection function ε_X for the set X, define a selection function $\varepsilon_{f(X)}$ for the set $f(X)$ by $\varepsilon_{f(X)}(p) = f(\varepsilon_X(p \circ f))$. ⊣

For more details and more closure properties of searchable sets, the interested reader can consult the above references.

THEOREM 10.5. *If a sequence of sets $X_n \subseteq 2^{\mathbb{N}}$ have selection functions ε_{X_n}, then the set $Y = \overline{\sum}_n X_n$ has a selection function ε_Y.*

PROOF. Define

$$\varepsilon_Y(p)(n) = 0 \iff \exists k \le n(1^k 0 \sqsubseteq (m \mapsto p(\varepsilon_{1^m 0 X_m}(p))) \wedge \varepsilon_{1^k 0 X_k}(p)(n) = 0).$$

By Section 2, the condition $1^k 0 \sqsubseteq (m \mapsto p(\varepsilon_{1^m 0 X_m}(p)))$ holds iff p has a root in $1^k 0 X_k$ but not in $1^m 0 X_m$ for $m < k$, and in this case $\varepsilon_{1^k 0 X_k}(p)$ is a root. Hence

$$1^k \sqsubseteq \varepsilon_Y(p) \iff \forall m < k(\forall \alpha \in 1^m 0 X_m(p(\alpha) = 1), \tag{3}$$

$$1^k 0 \sqsubseteq \varepsilon_Y(p) \implies p(\varepsilon_{1^k 0 X_k}(p)) = 0, \tag{4}$$

$$1^k 0 \sqsubseteq \varepsilon_Y(p) \implies \varepsilon_Y(p) = \varepsilon_{1^k 0 X_k}(p), \tag{5}$$

$$\varepsilon_Y(p) = \infty \iff \forall n \in \mathbb{N} \forall \alpha \in 1^n 0 X_n(p(\alpha) = 1). \tag{6}$$

To prove that ε_Y is a selection function for Y, let $p\colon \mathbb{N}_\infty \to 2$ be a map and assume that $p(\varepsilon_Y(p)) = 1$. For any k, if $1^k 0 \sqsubseteq \varepsilon_Y(p)$ then $\varepsilon_Y(p) = \varepsilon_{1^k 0 X_k}(p)$ by (5) and hence $p(\varepsilon_{1^k 0 X_k}(p)) = 1$ by the assumption and the extensionality of p, which contradicts (4). Hence $1^k 0 \not\sqsubseteq \varepsilon_Y(p)$, and so $\varepsilon_Y(p) = \infty$ by Lemma 10.1. It follows that $p(\infty) = 1$, by the assumption and the extensionality of p. Because $\bigcup_n 1^n 0 X_n \cup \{\infty\}$ is a full subset of Y by Lemma 10.2, we conclude that $p(\alpha) = 1$ for any $\alpha \in Y$ by (6) and Lemma 3.4, which shows that ε_Y is a selection function for the set Y. ⊣

Say that ε is an inf-*selection* if $\varepsilon(p)$ is the infimum of the set of roots of p for any map p. Notice that if X has an inf-selection then every decidable subset of X has a supremum, given by the infimum of its complement. It is straightforward that if ε_{X_n} is an inf-selection then so is $\varepsilon_{1^n 0 x_n}$. Continuing from Theorem 10.5:

LEMMA 10.6. *If each ε_{X_n} is an* inf*-selection, then so is ε_Y.*

PROOF. With the same general argument of Lemma 9.3, for any lower bound y of the set of roots, $y \le \varepsilon_Y(p)$. In order to show that $\varepsilon_Y(p) \le \alpha$ for any given root $\alpha \in Y$, assume that $\alpha < \varepsilon_Y(p)$. Let $s \in 2^*$ with $s0 \sqsubseteq \alpha$ and $s1 \sqsubseteq \varepsilon_Y(p)$. If $s = 1^n$ where n is the length of s, then $\alpha \in 1^n 0 X_n$ by definition of Y, but then $p(\alpha) = 1$ by (3) with $k = n + 1$, which contradicts the assumption that α is a root. Hence s is of the form $1^k 0 t$ for a unique $(k, t) \in \mathbb{N} \times 2^*$, and $1^k 0$ is a prefix of both α and $\varepsilon_Y(p)$. By definition of Y, we have that $\alpha \in 1^k 0 X_k$, and $\varepsilon_{1^k 0 X_k}(p)$ is a root by (4). Because $\varepsilon_{1^k 0 X_k}$ is an inf-selection and α is a root in $\alpha \in 1^k 0 X_k$, we conclude that $\varepsilon_{1^k 0 X_k}(p) \le \alpha$, which contradicts the assumption $\alpha < \varepsilon_Y(p)$ because $\varepsilon_Y(p) = \varepsilon_{1^k 0 X_k}(p)$ by (5). Hence we conclude that $\varepsilon_Y(p) \le \alpha$ by definition of $\le$. ⊣

LEMMA 10.7.
1. *If each X_n has a (countable) full subset F_n, then $\bigcup_n 1^n 0 F_n \cup \{\infty\}$ is a (countable) full subset of $\overline{\sum}_n X_n$.*
2. *If each F_n satisfies transfinite induction for arbitrary predicates, then does $\bigcup_n 1^n 0 F_n \cup \{\infty\}$.*

PROOF. Fullness follows from Lemma 10.2. Given enumerations $e_n\colon \mathbb{N} \to F_n$, define $e\colon \mathbb{N} \to \bigcup_n 1^n 0 F_n \cup \{\infty\}$ by $e(0) = \infty$ and $e(\langle m, n\rangle + 1) = e_m(n)$

where $(m,n) \mapsto \langle m,n\rangle \colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is a pairing function. The preservation of transfinite induction is trivial using induction, as the sets are mutually disjoint, including $\{\infty\}$. ⊣

Corollary 10.8. *The squashed sum of countably many ordinals is an ordinal.*

And the iteration of squashed sums produces ordinals with countable full subsets.

Alternative construction and technical discussion. For the sake of mathematical economy, because $\mathbb{N}_\infty$ arises as the squashed sum of singletons, we could have omitted the proofs of the main Theorems 3.5 and 9.4. Conversely, Theorem 10.5 can also be naturally derived from Theorem 3.5 as follows. Given a sequence of searchable sets $X_n \subseteq 2^{\mathbb{N}}$, we can define an $\mathbb{N}_\infty$-indexed family of searchable sets

$$Y_x \subseteq 2^{\mathbb{N}}$$

such that $Y_{\underline{n}} = 1^n 0 X_n$ and $Y_\infty = \{\infty\}$, and, moreover,

$$Y \stackrel{\text{def}}{=} \overline{\sum_x} X_n = \bigcup_{x\in\mathbb{N}_\infty} Y_x,$$

so that search over Y can be reduced to search over $\mathbb{N}_\infty$:

$$Y_x = \{\alpha \in 2^{\mathbb{N}} \mid \forall n \in \mathbb{N}(x = \underline{n} \Longrightarrow \alpha \in 1^n 0 X_n)\},$$
$$\varepsilon_{Y_x}(p)(n) = 0 \iff \exists k \le n(x = \underline{k} \wedge \varepsilon_{1^k 0 X_k}(p)(n) = 0).$$

To conclude, we use:

Lemma 10.9. *If a set X has a selection function ε_X and $Y_x \subseteq Z$ is an X-indexed family of sets with selection functions ε_{Y_x}, then the set $Y = \bigcup_{x\in X} Y_x$ has a selection function ε_Y given by*

$$\varepsilon_Y(p) = \varepsilon_{Y_x}(p), \text{ where } x = \varepsilon_X(x \mapsto p(\varepsilon_{Y_x}(p))).$$

This lemma describes a higher-type functional $(X \to JZ) \to (JX \to JZ)$, which amounts to the (internal) Kleisli extension operation of the selection monad J discussed in [14].

The selection function of $\mathbb{N}_\infty$ satisfies the equation

$$\varepsilon(p) = \begin{cases} 0, & \text{if } p(0) = 0,\\ \varepsilon(x \mapsto p(x+1)) + 1, & \text{otherwise.}\end{cases}$$

This amounts to a definition by co-recursion, using the fact that $\mathbb{N}_\infty$ is the final co-algebra of the functor $X \mapsto 1 + X$. The definitions of ε_Y as in Theorem 10.5 and of ε_{Y_x} can also be naturally written in co-recursive form, using the fact that $2^{\mathbb{N}}$ is the final co-algebra of the functor $X \mapsto 2 \times X$. One is left wondering whether, for the manifestation of ordinals discussed here, that arise by iterating successor and squashed sums, starting from one, suitable co-induction and co-recursion principles generalizing those of $\mathbb{N}_\infty$ would be more natural than induction and recursion. Because our ordinals are retracts of $2^{\mathbb{N}}$, the co-induction principle for $2^{\mathbb{N}}$ is in a way inherited by them, but this is not the whole story.

§11. Meta-mathematical discussion. Coquand, Hancock and Setzer [8] discuss how one can transfinitely iterate an $\mathbb{N}$-ary operation. For this purpose, they consider *limit structures*. A limit structure is a set X (which we will choose to be the Cantor space, or the space of selection functions $J2^{\mathbb{N}}$) together with an element $x \in X$ (we choose the ordinal one), a function $f\colon X \to X$ (we choose the successor function), and $l\colon (\mathbb{N} \to X) \to X$ (they think of the supremum of a sequence of ordinals, but we choose the squashed sum). They show how any limit structure can reach γ-transfinite iteration for any $\gamma < \epsilon_0$ in the presence of simple types, and how to go beyond in richer type systems. Thus, with classical eyes, one can see how to get these ordinals embedded in the Cantor space with our constructions. It remains to explore how to see this constructively, and how much ordinal arithmetic can be performed in the Cantor space.

In the model of continuous functionals of HA^ω, searchable sets satisfy a compactness condition [10], and this is why ω cannot be embedded within HA^ω as an ordinal in our sense in the Cantor space. It is not hard to see that our embedding of ordinals always produces successor ordinals, or closed ordinal intervals $[0, \gamma]$, with the interval topology, which are compact as they should. The ordinals 2 and $\omega + 1$ can be embedded, but the ordinal $2^{\omega+1} = 2^\omega \times 2 = \omega \times 2$ cannot, because it is not a successor ordinal. Hence there is no hope of performing exponentiation for our notion of ordinal, but this argument does not preclude the possibility of defining a construction that dominates exponentiation.

The argument of the first paragraph shows that it is not possible to embed ϵ_0 in $2^{\mathbb{N}}$ using limit structures. However, in principle it would be possible to embed $[0, \epsilon_0]$ and higher ordinals intervals in $2^{\mathbb{N}}$ within HA^ω without contradicting the fact that limit structures can only reach ordinals below ϵ_0. We do not dare to conjecture this or its negation, because we do not know enough about ordinals embedded in the Cantor space, in the sense of Section 9, other than they are closed under successor and squashed sums, they are searchable, they are retracts of the Cantor space, and have countable full subsets.

We conjecture that only sets with countable full subsets can be proved to be searchable in HA^ω. Assuming Brouwerian axioms, it is well known that the Cantor space is searchable, but the Cantor space cannot be proved to be searchable in HA^ω [12]. Because, classically, any uncountable compact subset of the Cantor space has a copy of the Cantor space, it is plausible that a construction of a searchable set with no countable full subset allows the construction of a selection function for the Cantor space. We leave this as an open problem, and this time we do dare to conjecture this.

A (technically justified) philosophical conclusion of this investigation is that it is misguided to assume that only the finite sets can satisfy the principle of omniscience in spartan constructive mathematics. We have shown that arbitrarily complex, infinite examples can be constructed. As far as we know, this is new. The reason $\mathbb{N}$ fails to provably satisfy the principle of omniscience in all varieties of constructive mathematics is subtler and deeper than the mere fact that it is infinite. There must be many more, seemingly unlikely or counter-intuitive or surprising, and hopefully useful, instances of excluded middle to be found in spartan constructive mathematics.

§12. Related work and acknowledgments. The investigation reported here has its origin in Exercise 1 of [1, page 581], attributed to Kreisel, which was brought to our attention by Gordon Plotkin and Alex Simpson, and which we found to be intimately connected with our own work [9, 10]. Dag Normann directed us to Grilliot's Trick [15, 16]. Thierry Coquand pointed out that Brouwer's 1927 proof of Theorem 1 of [22, page 459] can be seen as a forerunner of Kreisel's Trick, and Douglas Bridges mentioned Bishop's 1967 proof of Lemma 7 of [3, page 177]. John Longley mentioned the original proof of the Kreisel–Lacombe–Shoenfield theorem [19], although the idea is perhaps somewhat obscured there by several other ingredients. This is compatible with the clean view of this theorem offered by Ishihara [17].

In [22, page 459], [3, page 177], [17], the authors build a Cauchy sequence from an infinite binary sequence in order to perform a feat that amounts to a constructively valid instance of Markov's principle without assuming the principle, or, more ambitiously, a highly suspicious, but again constructively valid, instance of LPO. In the KLS theorem a sequence is constructed in a similar manner, but MP (or its weak version WMP [17]) is invoked in a step of the proof. All these constructions of Cauchy sequences from binary sequences amount to the fact that $\mathbb{N}_\infty$ is the generic convergent sequence.

We warmly thank the people that provided the above references, and also Thierry Coquand, Hajime Ishihara, John Longley, and Dag Normann for fruitful discussions.

References

[1] H.P. Barendregt, ***The lambda-calculus: its syntax and semantics***, North-Holland, 1984.

[2] M.J. Beeson, ***Foundations of constructive mathematics***, Springer, 1985.

[3] E. Bishop, ***Foundations of constructive analysis***, McGraw-Hill Book Co., New York, 1967.

[4] A. Bove and P. Dybjer, *Dependent types at work*, ***Proceedings of Language Engineering and Rigorous Software Development, LNCS***, vol. 5520 (2009), pp. 57–99.

[5] D. Bridges and F. Richman, ***Varieties of constructive mathematics***, London Mathematical Society Lecture Note Series, vol. 97, Cambridge University Press, Cambridge, 1987.

[6] D. Bridges, D. van Dalen, and H. Ishihara, *Ishihara's proof technique in constructive analysis*, ***Indagationes Mathematicae***, vol. 14 (2003), no. 2, pp. 163–168.

[7] D. Bridges and L. Vîţă, *A general constructive proof technique*, ***Electronic Notes in Theoretical Computer Science***, vol. 120 (2005), pp. 31–43.

[8] T. Coquand, P. Hancock, and A. Setzer, *Ordinals in type theory*, Invited talk at Computer Science Logic (CSL), Aarhus, `http://www.cse.chalmers.se/~coquand/ordinal.ps`, August 1997.

[9] M. H. Escardó, *Infinite sets that admit fast exhaustive search*, ***Proceedings of the 22nd annual ieee symposium on logic in computer science***, IEEE Computer Society, 2007, pp. 443–452.

[10] M. H. Escardó, *Exhaustible sets in higher-type computation*, ***Logical Methods in Computer Science***, vol. 4 (2008), no. 3, pp. 3:3, 37.

[11] M. H. Escardó, *Infinite sets that satisfy the principle of omniscience in all varieties of constructive mathematics*, Martin-Löf formalization, in Agda notation, of part of the paper with the same title. University of Birmingham, UK, `http://www.cs.bham.ac.uk/~mhe/papers/omniscient/AnInfiniteOmniscientSet.html`, September 2011.

[12] M. H. Escardó and P. Oliva, *Bar recursion and products of selection functions*, Available from the authors' web page, November 2010.

[13] ———, *Searchable sets, Dubuc–Penon compactness, omniscience principles, and the Drinker Paradox*, ***Computability in europe 2010, abstract and handout booklet*** (F. Ferreira, H. Guerra, E. Mayordomo, and J. Rasga, editors), Centre fo Applied Mathematics and Information Technology, Department of Mathematics, University of Azores, 2010, pp. 168–177.

[14] ———, *Selection functions, bar recursion and backward induction*, ***Mathematical Structures in Computer Science. A Journal in the Applications of Categorical, Algebraic and Geometric Methods in Computer Science***, vol. 20 (2010).

[15] T. J. Grilliot, *On effectively discontinuous type-2 objects*, ***The Journal of Symbolic Logic***, vol. 36 (1971), pp. 245–248.

[16] J. P. Hartley, *Effective discontinuity and a characterisation of the superjump*, ***The Journal of Symbolic Logic***, vol. 50 (1985), no. 2, pp. 349–358.

[17] H. Ishihara, *Continuity and nondiscontinuity in constructive mathematics*, ***The Journal of Symbolic Logic***, vol. 56 (1991), no. 4, pp. 1349–1354.

[18] ———, *Constructive reverse mathematics: compactness properties*, ***From sets and types to topology and analysis***, Oxford Logic Guides, vol. 48, Oxford Univ. Press, Oxford, 2005, pp. 245–267.

[19] G. Kreisel, D. Lacombe, and J. R. Shoenfield, *Partial recursive functionals and effective operations*, ***Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting)***, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam, 1957, pp. 290–297.

[20] E. Palmgren, *Constructivist and structuralist foundations: Bishop's and Lawvere's theories of sets*, to appear in *Annals of Pure and Applied Logic*, available at the author's web page, 2011.

[21] A. S. Troelstra and D. van Dalen, ***Constructivism in mathematics. an introduction***, Studies in Logic and the Foundations of Mathematics, vol. 121 and 123, North Holland, Amsterdam, 1988.

[22] J. van Heijenoort, ***From Frege to Gödel : A Source Book in Mathematical Logic, 1879-1931***, Harvard University Press, 1967.

SCHOOL OF COMPUTER SCIENCE, UNIVERSITY OF BIRMINGHAM, B15 2TT, UK