Infinite sets that satisfy the principle of omniscience in all varieties of constructive mathematics

(Draft for a talk at Types'2011, version of July 15, 2011, with minor updates September 28, 2011)

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Abstract

In a minimalistic setting for constructive mathematics, without assuming Brouwerian axioms such as continuity, bar induction or fan theorem, we show that there are plenty of infinite sets that satisfy the omniscience principle.

1 Introduction

We show that there are plenty of infinite sets X that satisfy the omniscience principle

$$\forall \text{ extensional } p\colon X \to 2(\exists x \in X(p(x) = 0) \vee \forall x \in X(p(x) = 1)).$$

We do this in a spartan setting for constructive/intuitionistic mathematics [4, 2, 18] that is compatible with classical mathematics and *doesn't* postulate or reject contentious axioms or principles such as choice, powerset, Markov's principle, double-negation shift, continuity, bar induction, fan theorem, or Church's thesis. We don't postulate extensionality, but we assume it as a hypothesis of definitions or formulations theorems when our technical development demands it, as in this example. With this, we account for intensional type theory, in particular, and we make explicit that the development does crucially rely on extensionality. We need some amount of higher types, at least level 3 to develop the main construction that allows to build increasingly more complex omniscient sets of type 1. With the help of a further construction [6], the higher we climb the type hierarchy the more complex omniscient sets of type 1 we get (see the last paragraph of this section).

Our development makes sense in Bishop mathematics and Brouwer and Russian intuitionism, among others, and can be formalized in e.g. HA^ω, intensional Martin-Löf type theory, CZF, the internal language of the free topos with natural numbers object [17], or indeed any variety of formalized classical set theory. But everything we say is a triviality from the point of view of classical mathematics, although probably unbelievable at first sight from the point of view of minimalistic constructive mathematics, despite the fact that Grilliot [13], Ishihara [15] and others have established similarly outrageous facts, with similar constructions.

A first example of an omniscient set is the *one-point compactification of the natural numbers*,

$$\mathbb{N}_\infty = \{x \in 2^{\mathbb{N}} \mid \forall i \in \mathbb{N}(x_i \geq x_{i+1})\},$$

also known as the *generic convergent sequence*. We generalize this to many other infinite subsets of the Cantor space $2^{\mathbb{N}}$. These subsets turn out to be ordinals in a constructive sense, with respect to the lexicographic order, satisfying both a well-foundedness condition with respect to decidable subsets, and transfinite induction restricted to decidable predicates. The use of higher types allows us to reach any ordinal below ϵ_0, and stronger systems allow us to get higher, using [6].

2 Selection of roots of 2-valued functions

The omniscience principle for a set X says that it is decidable whether a given extensional $p\colon X \to 2$ has a root. For some sets X, given any extensional $p\colon X \to 2$, it is possible to construct a putative root $x \in X$ such that p has a root if and only if x is indeed a root. The simplest, non-trivial, and perhaps surprising example of such a set X is given in Section 3, followed by increasingly more complex examples in Section 7. In this section we briefly investigate general aspects of this phenomenon and its relation to omniscience, to be exploited in those sections.

A *selection function* for a set X is a functional $\varepsilon\colon (X \to 2) \to X$ such that for all extensional $p\colon X \to 2$,

$$p(\varepsilon(p)) = 1 \implies \forall x \in X(p(x) = 1).$$

This says that if $\varepsilon(p)$ is not a root, then no $x \in X$ is a root. We call a set *searchable* if it has a selection function. (The axiom of choice implies that any omniscient, inhabited set is searchable [9], if we restrict the domain of ε to extensional functions. But, as discussed in the introduction, we are not assuming extensionality or choice. Moreover, here we are not restricting ε in such a way, although this is unimportant.)

Lemma 2.1 *Any searchable set satisfies the omniscience principle.*

Proof This follows from the definition of selection function, using the facts that $p(\varepsilon(p)) = 0$ implies $\exists x \in X(p(x) = 0)$ by considering $x = \varepsilon(p)$, and that either $p(\varepsilon(p)) = 0$ or else $p(\varepsilon(p)) = 1$ by the decidability of equality of the set $2 = \{0, 1\}$ of binary numbers. □

Moreover, assuming a selection function:

Lemma 2.2 *An extensional p has a root if and only if $\varepsilon(p)$ is a root.*

Proof If p has a root x, that is, $p(x) = 0$, then $\neg\forall x \in X(p(x) = 1)$, and hence the contra-positive of the definition of selection function gives $p(\varepsilon(p)) \neq 1$, and therefore $p(\varepsilon(p)) = 0$, which shows that $\varepsilon(p)$ is a root. □

A similar argument shows that $\neg\neg\exists x \in X(p(x) = 0) \implies \exists x \in X(p(x) = 0)$ in the presence of a selection function. Lemma 2.2 shows that if X is searchable, there is a functional $E_X\colon (X \to 2) \to 2$ whose extensional roots p are the functions that have roots,

$$E_X(p) = 0 \iff \exists x \in X(p(x) = 0),$$

constructed as

$$E_X(p) = p(\varepsilon(p)).$$

Proposition 2.3 *If Y is searchable, the choice principle*

$$(\forall x \in X \exists y \in Y(A(x, y))) \implies \exists f\colon X \to Y \forall x \in X(A(x, f(x)))$$

holds for any set X and any extensional, decidable propositional function $A(x, y)$.

Proof Define $f(x) = \varepsilon_Y(y \mapsto a(x, y))$ where $a\colon X \times Y \to 2$ is the function whose roots are the pairs related by A, which exists by the decidability of $A(x, y)$. □

The notion of selection function is investigated in [7, 10, 9] from various points of view, sometimes switching the roles of 0 and 1, but this is of course unimportant as is simply a matter of naming conventions. Sets that satisfy the omniscience principle are called *exhaustible* in [7], where they are studied in the context of classical higher-type computability theory with partial continuous functionals. Here our reasoning is purely constructive, under a minimalistic foundation for constructive mathematics, where all functions are tacitly total and not assumed to be continuous.

3 Searchability of the generic convergent sequence

The set $\mathbb{N}_\infty$ defined in the introduction has elements

$$\underline{n} = 1^n 0^\omega, \qquad \infty = 1^\omega,$$

defined in more conventional form by $\underline{n}_i = 1 \iff i < n$ and $\infty_i = 1$. We work with extensionally defined equality on the Cantor space $2^\mathbb{N}$ and hence on its subset $\mathbb{N}_\infty$. (If we are working within e.g. intensional type theory, then this will be a potentially different (finner) equality from that which comes with the underlying formal system. If we are working in HA^ω, this will be our definition of equality on the Cantor space.)

Proposition 3.1 *The set $\mathbb{N}_\infty$ is a retract of $2^\mathbb{N}$ in the sense that there is an idempotent map $r\colon 2^\mathbb{N} \to 2^\mathbb{N}$ such that $\alpha \in 2^\mathbb{N}$ is in $\mathbb{N}_\infty$ if and only if $r(\alpha) = \alpha$.*

Proof For example, we can let $r(\alpha)(n) = \min\{\alpha_k \mid k \leq n\}$. □

Lemma 3.2 *For any $x \in \mathbb{N}_\infty$, if $x_n = 0$ then $x = \underline{k}$ for some $k \leq n$.*

Proof If $x_j = 0$ for all $j < n$, take $k = 0$. Otherwise take the largest $j < n$ with $x_j = 1$ and let $k = j + 1$. □

Lemma 3.3 $\forall x \in \mathbb{N}_\infty\, (\forall k \in \mathbb{N}(x \neq \underline{k})) \implies x = \infty$.

Proof Let $x \in \mathbb{N}_\infty$, assume that $\forall k \in \mathbb{N}(x \neq \underline{k})$, and let $n \in \mathbb{N}$. If $x_n = 0$, then $x = \underline{k}$ for some k by Lemma 3.2, which contradicts the hypothesis, and hence we must have $x_n = 1$. Because n is arbitrary, $x = \infty$. □

We say that a subset F of a set X is *full* if its complement is empty. Because we are not assuming the axiom of extensionality, we take the complement to be

$$X \setminus F = \{x \in X \mid \forall x' \in F(x \neq x')\}.$$

By Lemma 3.3 with $\underline{\mathbb{N}} = \{\underline{n} \mid n \in \mathbb{N}\}$:

Lemma 3.4 *$\underline{\mathbb{N}} \cup \{\infty\}$ is a countable, full subset of $\mathbb{N}_\infty$.*

Notice that $\mathbb{N}_\infty \subseteq \underline{\mathbb{N}} \cup \{\infty\}$ implies LPO, that is, the omniscience of $\mathbb{N}$, and so does the denumerability of $\mathbb{N}_\infty$. Of course, we are not postulating or rejecting LPO, which is and will remain a taboo for constructive mathematics.

Lemma 3.5 (Full subsets are dense) *Any extensional $p\colon X \to 2$ that is constant on a full subset is constant on the whole of X.*

Proof Assume that p is constant on a full subset $F \subseteq X$ with value $y \in 2$. If we had $p(x) \neq y$ for $x \in X$, the extensionality of p would give $x \neq x'$ for every $x' \in F$, which contradicts the fullness of F. □

Theorem 3.6 *The set $\mathbb{N}_\infty \subseteq 2^\mathbb{N}$ has a selection function given by*

$$\varepsilon_{\mathbb{N}_\infty}(p)(n) = \min\{p(\underline{k}) \mid k \leq n\}.$$

Proof This is clearly well defined. (Notice that, omitting the subscript $\mathbb{N}_\infty$, we have that $\varepsilon(p) = r(n \mapsto p(\underline{n}))$, where r is the retraction defined in Proposition 3.1.) As $\varepsilon(p)(n) = 0$ iff $\exists k \leq n\, (p(\underline{k}) = 0)$, we have that

$$\varepsilon(p) = \underline{n} \iff p(\underline{n}) = 0 \wedge \forall k < n(p(\underline{k}) = 1), \tag{1}$$

$$\varepsilon(p) = \infty \iff \forall n \in \mathbb{N}(p(\underline{n}) = 1). \tag{2}$$

To prove that ε is a selection function for $\mathbb{N}_\infty$, let $p\colon \mathbb{N}_\infty \to 2$ be extensional and assume that $p(\varepsilon(p)) = 1$. Then $\varepsilon(p) \neq \underline{n}$ for any $n \in \mathbb{N}$, for if we had $\varepsilon(p) = \underline{n}$ we would have $p(\underline{n}) = 1$ by the assumption and the extensionality of p, which contradicts (1). Hence $\varepsilon(p) = \infty$ by Lemma 3.3. This implies both $p(\infty) = 1$, by the assumption and the extensionality of p, and $p(\underline{n}) = 1$ for every $n \in \mathbb{N}$, by (2). Therefore $p(x) = 1$ for any $x \in \mathbb{N}_\infty$ by Lemmas 3.4 and 3.5. □

4 Fundamental properties of omniscient sets

Call a set discrete if it has decidable equality. Regarding Theorem 4.1(4), notions of order will be discussed in more detail in Sections 5 and 6.

Theorem 4.1 (General properties)

1. *If X is omniscient, then $\neg\neg\exists x \in X(p(x) = 0) \implies \exists x \in X(p(x) = 0)$ for any extensional $p\colon X \to 2$.*
2. *If X is omniscient and Y is discrete, then the extensional functions $X \to Y$ under extensional equality form a discrete set.*
3. *If X and Y are omniscient, then for any extensional $q\colon X \times Y \to 2$,*
$$\exists x \in X\, \forall y \in Y(q(x,y) = 1) \vee \forall x \in X\, \exists y \in Y(q(x,y) = 0).$$
4. *If X is an omniscient ordered set with binary maximum $\max\colon X \times X \to X$, then for any extensional $p\colon X \to 2$,*
$$\exists x \in X\, \forall y \geq x(p(y) = 1) \vee \forall x \in X\, \exists y \geq x(p(y) = 0).$$

Proof (1): One always has $\neg\neg\exists x \in X(p(x) = 0) \implies \neg\forall x \in X(p(x) = 1)$, and the implication $\neg\forall x \in X(p(x) = 1) \implies \exists x \in X(p(x) = 0)$ follows directly from the definition of omniscience of X.

(2): $f = g$ iff $\forall x \in X(f(x) = g(x))$, and hence the claim follows by the omniscience of X applied to the extensional map $p\colon X \to 2$ defined by $p(x) = 0 \iff f(x) = g(x)$ using the discreteness of Y.

(3): By the omniscience of Y, there is an extensional predicate $p\colon X \to 2$ such that $p(x) = 1$ if $\exists y \in Y(q(x,y) = 0)$, and $p(x) = 0$ if $\forall y \in Y(q(x,y) = 1)$, and hence the result follows by the omniscience of X applied to p.

(4): This is an instance of (3) with $q(x,y) = p(\max(x,y))$. □

Theorem 4.2 (Closure properties)

1. *If X and Y are omniscient, then so is $X \times Y$.*
2. *If X is omniscient then so is its image $f(X)$ for any extensional $f\colon X \to Y$.*
3. *If X is omniscient then so is the union $\bigcup_{x\in X} Y_x$ of any X-indexed family of omniscient sets $Y_x \subseteq Z$.*
4. *If X is omniscient then so is any extensional, decidable subset $A \subseteq X$.*

Proof (1): The argument is the same as that of Theorem 4.1(3), using the fact that $\exists z \in X \times Y(q(z) = 0)$ is equivalent to $\exists x \in X \exists y \in X(q(x,y) = 0)$, but considering the function $p(x) = 0$ iff $\exists y \in Y(q(x,y) = 0)$ instead.

(2): $\exists y \in f(X)(q(y) = 0)$ is equivalent to $\exists x \in X(p(x) = 0)$ where $p(x) = q(f(x))$.

(3): $\exists z \in \bigcup_{x\in X} Y_x(q(x,y) = 0)$ is equivalent to $\exists x \in X \exists y \in Y_x(q(x,y) = 0)$.

(4): Given $p\colon A \to 2$, extend it to X by mapping all $x \notin A$ to 1, and apply the omniscience of X. □

Some (closure) properties of searchable sets from [7, 10] will be recalled in later sections as we need them for our technical development.

5 Ordinals

For the purposes of this investigation, we define the notion of *ordinal* by considering suitable formulations of well-ordering and transfinite induction that involve *decidability* (and extensionality) conditions. This notion of ordinal classically coincides with the classical one, but, from the point of view of constructive mathematics, it is probably at the same time audacious and restrictive, and hence not suitable as a substitute of more standard constructive notions of (countable) ordinal encodings. Nevertheless, in the following sections we will show that $\mathbb{N}_\infty$ and plenty of other subsets of the Cantor space are (omniscient) ordinals in our sense, under the lexicographic ordering.

Regarding the following definition, we are not able to show that a well-ordered set satisfies the principle of transfinite induction, but we won't be surprised if the conclusion is found to hold under additional, classically vacuous, assumptions, or even without assumptions in the Cantor space. For the moment we live with a pinch of salt:

Definition 5.1 (Ordinal) We take an *ordinal* to be a linearly ordered set that is well-ordered with respect to *decidable* subsets, in the sense that every inhabited, decidable, extensional subset has a least element, and satisfies the usual principle of transfinite induction, restricted to *decidable* extensional predicates.

We don't assume that our constructive setting allows us to quantify over the subsets of a set or over predicates, but this definition is not problematic because the two conditions amount to

$$\begin{aligned}&\forall \text{extensional } p\colon X \to 2,\\ &\quad (\exists y \in X(p(y) = 0)) \implies \exists x \in X(p(x) = 0 \land \forall y \in X(p(y) = 0 \implies x \le y)),\\ &\quad (\forall x \in X(\forall y < x(p(y) = 0)) \implies p(x) = 0) \implies \forall x \in X(p(x) = 0)),\end{aligned}$$

and we do tacitly assume that we can quantify over functions. By Lemma 3.5:

Lemma 5.2 *If a linearly ordered set X has a full subset that satisfies transfinite induction for arbitrary predicates, then X satisfies transfinite induction for decidable extensional predicates.*

For the sake of completeness, in the remainder of this section we recall the details of the standard definition of linear order that has proved to be suitable for constructive reasoning [18]. A *linear order* on a set X is a binary relation $<$ satisfying

$$x < y \land y < z \implies x < z, \; \neg(x < y \lor y < x) \iff x = y, \; x < y \implies x < z \lor z < y.$$

These conditions imply that $x \not< x$ and that the relation defined by

$$x \sharp y \iff x < y \lor y < x$$

is a tight apartness relation (the *intrinsic apartness*). Recall that this means that

$$\neg(x \sharp x), \qquad x \sharp y \implies y \sharp x, \qquad x \sharp y \implies x \sharp z \lor z \sharp y, \quad \neg(x \sharp y) \implies x = y,$$

where the last condition is called *tightness.* The axioms for a linear order also imply that the relation defined by

$$x \le y \iff y \not< x$$

is a partial order (the *intrinsic partial order*), that is, it is reflexive, transitive, and antisymmetric. Moreover, it satisfies

$$x < y \le z \implies x < z, \quad x \le y < z \implies x < z.$$

Notice that the negatively defined partial order $\le$ is positively characterized as

$$x \le y \iff (x \sharp y \implies x < y) \iff (y < x \implies x < y).$$

6 Ordinals in the Cantor space

The *lexicographic order* of $2^{\mathbb{N}}$ is $\alpha < \beta \iff \exists n \in \mathbb{N}(\forall i < n(\alpha_i = \beta_i) \wedge \alpha_n < \beta_n)$, or, equivalently,

$$\alpha < \beta \iff \exists s \in 2^*(s0 \sqsubseteq \alpha \wedge s1 \sqsubseteq \beta),$$

where 2^* is the set of finite sequences and $\sqsubseteq$ denotes the prefix relation. It is easy to see that this is a linear order, whose intrinsic apartness coincides with the standard apartness on the Cantor space,

$$\alpha \sharp \beta \iff \exists n \in \mathbb{N}(\alpha_n \neq \beta_n).$$

Notice that the pointwise and lexicographic orders agree on $\mathbb{N}_\infty$ (and hence binary minimum and maximum in $\mathbb{N}_\infty$ can be calculated pointwise). Moreover, for $x, y \in \mathbb{N}_\infty$, we have that $\underline{n} < x \iff x_n = 1$ and $x \leq \underline{n} \iff x_n = 0$, and so

$$x < y \iff \exists n(x_n = 0 \wedge y_n = 1) \iff \exists n(x \leq \underline{n} < y).$$

Also $\underline{m} < \underline{n}$ in $\mathbb{N}_\infty$ iff $m < n$ in $\mathbb{N}$, and $\underline{n} < \infty$ for every $n \in \mathbb{N}$.

To build finite ordinals in the Cantor space, we start from $0 = \emptyset$ and $1 = \{\underline{0}\}$, and for $X, Y \subseteq 2^{\mathbb{N}}$ we define $X + Y = 0X \cup 1Y$ by prefixing 0 and 1 to the elements of X and Y. The ordinal 0 is anomalous from the point of view of this investigation, because it is not searchable. But 1 is (with a unique selection function), and if X and Y are searchable then so is $X + Y$, and the construction can be performed so that if the selection functions for X and Y calculate infima of sets of roots, then so does that for $X + Y$, and hence we get all the finite ordinals embedded into $2^{\mathbb{N}}$.

We remark that we will not be able to embed into the Cantor space an ordinal that classically is ω (for the reasons discussed in Section 8). In fact, with classical eyes, we will be able to account for successor ordinals only, or equivalently ordinal intervals of the form $[0, \gamma]$. From this point of view, the situation for 0 is no longer anomalous, as this naturally excludes the empty set, even when $\gamma = 0$. Moreover, still with classical eyes, we'll be able to account for countable ordinals only. The reason is that only countable ordinals can be embedded in the natural order of the real line, as is well known, and the Cantor space (continuously) order-embeds into the real line via Cantor's third-middle construction. Of course, $\mathbb{N}_\infty$ is $\omega + 1$, or equivalently $[0, \omega]$, with classical eyes. We now switch back to sharper eyes.

Lemma 6.1 *If $p \colon \mathbb{N}_\infty \to 2$ is extensional, then $\varepsilon_{\mathbb{N}_\infty}(p)$ is the infimum of the set of roots of p.*

Proof For any lower bound y of the set of roots, $y \leq \varepsilon(p)$: If $p(\varepsilon(p)) = 0$ then $\varepsilon(p)$ is in the set and we are done. Otherwise the claim is vacuous because the set is empty by definition of selection function, and so we are done with the first part of the proof. To conclude, we show that $\varepsilon(p)$ is a lower bound of the set of roots. In order to show that $\varepsilon(p) \leq x$ for any given root x, assume that $x < \varepsilon(p)$. By the definitions of ε and the order, there is $n \in \mathbb{N}$ such that $(i)\, x_n = 0$ and $(ii) \min\{p(\underline{k}) \mid k \leq n\} = 1$. By (i) and Lemma 3.2, there is $j \leq n$ with $x = \underline{j}$. By (ii) we have $p(\underline{k}) = 1$ for every $k \leq n$, and hence $p(x) = p(\underline{j}) = 1$ by the extensionality of p, which contradicts the fact that x is a root. Discharging the assumption $x < \varepsilon(p)$, we conclude that $\varepsilon(p) \leq x$. □

Theorem 6.2 *$\mathbb{N}_\infty$ is an ordinal.*

Proof The well-orderedness condition of Definition 5.1 follows from Lemma 6.1, because, taking Lemma 2.2 into account, it says that any extensional $p \colon \mathbb{N}_\infty \to 2$ has a least root if it has a root. The transfinite induction principle for extensional p follows by Lemmas 3.4 and 5.2, because the full subset $\underline{\mathbb{N}} \cup \{\infty\}$ satisfies transfinite induction for arbitrary predicates, by induction on $\mathbb{N}$ and case analysis. □

7 Squashed sums

Given countably many searchable sets $X_n \subseteq 2^{\mathbb{N}}$, we show that their *squashed sum* that arises as their disjoint union with an added point at infinity is also searchable. We construct the squashed sum as a subset of $2^{\mathbb{N}}$, so that we can (transfinitely) iterate this procedure. We rescale and translate each set X_n by prefixing the finite sequence $1^n 0$ to its members, in order to make the sets disjoint, and at the same time smaller in diameter and arbitrarily close to the sequence $\infty = 1^\omega$ as n increases. With classical eyes, the squashed sum will be $\bigcup_n 1^n 0 X_n \cup \{\infty\}$, and if the sets X_n are ordinals then the squashed sum will be the successor of their ordinal sum. Constructively, it will be the case that the squashed sum of countably many ordinals is an ordinal, for ordinals in the sense of Section 5, which we denote by $[0, \sum_n X_n]$ or $\overline{\sum}_n X_n$. The set $\mathbb{N}_\infty$ will be the squashed sum of the constant sequence $X_n = \{\underline{0}\}$, so that the results of this section will subsume the theorems we formulated and proved for $\mathbb{N}_\infty$ in the previous sections (but see the discussion at the end of this section).

Lemma 7.1 $\forall \alpha \in 2^{\mathbb{N}} (\forall n \in \mathbb{N} (1^n 0 \not\sqsubseteq \alpha) \implies \alpha = \infty)$.

Proof For any k we have $\alpha_k = 1$, for if we had $\alpha_k = 0$ then we would have $1^n 0 \sqsubseteq \alpha$ for some $n \leq k$, which contradicts the hypothesis. □

For countably many given sets $X_n \subseteq 2^{\mathbb{N}}$, we define their squashed sum by

$$\overline{\sum_n} X_n = \{\alpha \in 2^{\mathbb{N}} \mid \forall n \in \mathbb{N} (1^n 0 \sqsubseteq \alpha \implies \alpha \in 1^n 0 X_n)\}.$$

Then $\infty \in \overline{\sum}_n X_n$ vacuously, and by construction $1^m 0 X_m \subseteq \overline{\sum}_n X_n$. Moreover:

Lemma 7.2 $\forall \alpha \in \overline{\sum}_n X_n (\forall n \in \mathbb{N} (\alpha \notin 1^n 0 X_n) \implies \alpha = \infty)$.

Proof For any n, if we had $1^n 0 \sqsubseteq \alpha$, then we would have $\alpha \in 1^n 0 X_n$ by definition of $\overline{\sum}_n X_n$, which contradicts the hypothesis. Hence the result follows from Lemma 7.1. □

The n-tail of a sequence $\alpha \in 2^{\mathbb{N}}$ is $\alpha^{(n)} \in 2^{\mathbb{N}}$ defined by $\alpha_i^{(n)} = \alpha_{n+i}$.

Proposition 7.3 *The retracts of the Cantor space are closed under the formation of squashed sums.*

Proof Given idempotent maps $r_n \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with image X_n, define $r \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $r(\alpha)(n) = 0 \iff \exists k \leq n (1^k 0 \sqsubseteq \alpha \wedge (1^k 0\, r_k(\alpha^{(k+1)}))_n = 0)$. □

To know that $1^n 0 X_n$ is searchable if X_n is, we use the following construction applied to the rescaling-translation map $\alpha \mapsto 1^n 0 \alpha$. This construction appears in [7, Proposition 4.3] and is studied in more generality in [10], which also shows it to be the functor of a certain selection monad J with $JX = ((X \to 2) \to X)$, where the resulting selection function $\varepsilon_{f(X)}$ amounts to $Jf(\varepsilon_X)$.

Lemma 7.4 *If X is searchable, then so is its image $f(X)$ for any extensional $f \colon X \to Y$.*

Proof Given a selection function ε_X for the set X, define a selection function $\varepsilon_{f(X)}$ for the set $f(X)$ by $\varepsilon_{f(X)}(p) = f(\varepsilon_X(p \circ f))$. □

For more details and more closure properties of searchable sets, the interested reader can consult the above references.

Theorem 7.5 *If a sequence of sets $X_n \subseteq 2^{\mathbb{N}}$ have selection functions ε_{X_n}, then the set $Y = \overline{\sum}_n X_n$ has a selection function ε_Y.*

Proof Define

$$\varepsilon_Y(p)(n) = 0 \iff \exists k \le n(1^k 0 \sqsubseteq (m \mapsto p(\varepsilon_{1^m 0 X_m}(p))) \ \wedge \ \varepsilon_{1^k 0 X_k}(p)(n) = 0).$$

By Section 2, the condition $1^k 0 \sqsubseteq (m \mapsto p(\varepsilon_{1^m 0 X_m}(p)))$ holds iff p has a root in $1^k 0 X_k$ but not in $1^m 0 X_m$ for $m < k$, and in this case $\varepsilon_{1^k 0 X_k}(p)$ is a root. Hence

$$\begin{aligned}
1^k \sqsubseteq \varepsilon_Y(p) &\iff \forall m < k(\forall \alpha \in 1^m 0 X_m(p(\alpha) = 1), && (3)\\
1^k 0 \sqsubseteq \varepsilon_Y(p) &\implies p(\varepsilon_{1^k 0 X_k}(p)) = 0, && (4)\\
1^k 0 \sqsubseteq \varepsilon_Y(p) &\implies \varepsilon_Y(p) = \varepsilon_{1^k 0 X_k}(p), && (5)\\
\varepsilon_Y(p) = \infty &\iff \forall n \in \mathbb{N} \forall \alpha \in 1^n 0 X_n(p(\alpha) = 1). && (6)
\end{aligned}$$

To prove that ε_Y is a selection function for Y, let $p \colon \mathbb{N}_\infty \to 2$ be extensional and assume that $p(\varepsilon_Y(p)) = 1$. For any k, if $1^k 0 \sqsubseteq \varepsilon_Y(p)$ then $\varepsilon_Y(p) = \varepsilon_{1^k 0 X_k}(p)$ by (5) and hence $p(\varepsilon_{1^k 0 X_k}(p)) = 1$ by the assumption and the extensionality of p, which contradicts (4). Hence $1^k 0 \not\sqsubseteq \varepsilon_Y(p)$, and so $\varepsilon_Y(p) = \infty$ by Lemma 7.1. It follows that $p(\infty) = 1$, by the assumption and the extensionality of p. Because $\bigcup_n 1^n 0 X_n \cup \{\infty\}$ is a full subset of Y by Lemma 7.2, we conclude that $p(\alpha) = 1$ for any $\alpha \in Y$ by (6) and Lemma 3.5, which shows that ε_Y is a selection function for the set Y. □

Say that ε is an inf-*selection* if $\varepsilon(p)$ is the infimum of the set of roots of p for any extensional p. Notice that if X has an inf-selection then every extensional decidable subset of X has a supremum, given by the infimum of its complement. It is straightforward that if ε_{X_n} is an inf-selection then so is $\varepsilon_{1^n 0 x_n}$. Continuing from Theorem 7.5:

Lemma 7.6 *If each ε_{X_n} is an* inf*-selection, then so is ε_Y.*

Proof With the same general argument of Lemma 6.1, for any lower bound y of the set of roots, $y \le \varepsilon_Y(p)$. In order to show that $\varepsilon_Y(p) \le \alpha$ for any given root $\alpha \in Y$, assume that $\alpha < \varepsilon_Y(p)$. Let $s \in 2^*$ with $s0 \sqsubseteq \alpha$ and $s1 \sqsubseteq \varepsilon_Y(p)$. If $s = 1^n$ where n is the length of s, then $\alpha \in 1^n 0 X_n$ by definition of Y, but then $p(\alpha) = 1$ by (3) with $k = n + 1$, which contradicts the assumption that α is a root. Hence s is of the form $1^k 0 t$ for a unique $(k, t) \in \mathbb{N} \times 2^*$, and $1^k 0$ is a prefix of both α and $\varepsilon_Y(p)$. By definition of Y, we have that $\alpha \in 1^k 0 X_k$, and $\varepsilon_{1^k 0 X_k}(p)$ is a root by (4). Because $\varepsilon_{1^k 0 X_k}$ is an inf-selection and α is a root in $\alpha \in 1^k 0 X_k$, we conclude that $\varepsilon_{1^k 0 X_k}(p) \le \alpha$, which contradicts the assumption $\alpha < \varepsilon_Y(p)$ because $\varepsilon_Y(p) = \varepsilon_{1^k 0 X_k}(p)$ by (5). Hence we conclude that $\varepsilon_Y(p) \le \alpha$ by definition of $\le$. □

Lemma 7.7

1. *If each X_n has a (countable) full subset F_n, then $\bigcup_n 1^n 0 F_n \cup \{\infty\}$ is a (countable) full subset of $\overline{\sum}_n X_n$.*
2. *If each F_n satisfies transfinite induction for arbitrary predicates, then does $\bigcup_n 1^n 0 F_n \cup \{\infty\}$.*

Proof Fullness follows from Lemma 7.2. Given enumerations $e_n \colon \mathbb{N} \to F_n$, define an enumeration $e \colon \mathbb{N} \to \bigcup_n 1^n 0 F_n \cup \{\infty\}$ by $e(0) = \infty$ and $e(\langle m, n \rangle + 1) = e_m(n)$ where $(m, n) \mapsto \langle m, n \rangle \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a pairing function. The preservation of transfinite induction is trivial as the sets are disjoint, including $\{\infty\}$. □

It is the second part that is relevant for the following corollary, but the first part shows that the (transfinite) iteration of the squashed sum produces ordinals with countable full subsets.

Corollary 7.8 *The squashed sum of countably many ordinals is an ordinal.*

Alternative construction and technical discussion. For the sake of mathematical economy, because $\mathbb{N}_\infty$ arises as the squashed sum of singletons, we could have omitted large parts of the previous sections, but the current exposition of the material is probably more digestible. More importantly, the theorems of this section can also be naturally derived from the previous sections, as follows. Given a sequence of searchable sets $X_n \subseteq 2^{\mathbb{N}}$, we can define an $\mathbb{N}_\infty$-indexed family of searchable sets

$$Y_x \subseteq 2^{\mathbb{N}}$$

such that

$$Y_{\underline{n}} = 1^n 0 X_n, \qquad Y_\infty = \{\infty\},$$

and, moreover,

$$Y \stackrel{\text{def}}{=} \overline{\sum_x} X_n = \bigcup_{x \in \mathbb{N}_\infty} Y_x,$$

so that search over Y can be reduced to search over $\mathbb{N}_\infty$ as constructed in the previous sections. In fact, define, for $x \in \mathbb{N}_\infty$,

$$Y_x = \{\alpha \in 2^{\mathbb{N}} \mid \forall n \in \mathbb{N}(x = \underline{n} \implies \alpha \in 1^n 0 X_n)\}$$

and

$$\varepsilon_{Y_x}(p)(n) = 0 \iff \exists k \leq n(x = \underline{k} \wedge \varepsilon_{1^k 0 X_k}(p)(n) = 0).$$

Notice that the condition $x = \underline{n}$ is decidable because it amounts to $1^n 0 \sqsubseteq x$. To conclude the alternative construction, we observe that:

Lemma 7.9 *If a set X has a selection function ε_X and $Y_x \subseteq Z$ is an X-indexed family of sets with selection functions ε_{Y_x}, then the set $Y = \bigcup_{x \in X} Y_x$ has a selection function ε_Y given by*

$$\varepsilon_Y(p) = \varepsilon_{Y_x}(p), \text{ where } x = \varepsilon_X(x \mapsto p(\varepsilon_{Y_x}(p))).$$

This lemma describes a higher-type functional $(X \to JZ) \to (JX \to JZ)$, which amounts to the (internal) Kleisli extension operation of the selection monad J discussed in [10].

In this sense, search over $\mathbb{N}_\infty$ can be seen as more fundamental than the preservation of searchability by the squashed sum construction. Hence we have decided to include both constructions (which concretely are easily seen to produce the same result), with an indication of how they can be reduced to each other, so that the readers can compare them and judge for themselves.

Notice that our selection function for $\mathbb{N}_\infty$ satisfies the equation

$$\varepsilon_{\mathbb{N}_\infty}(p) = \begin{cases} \underline{0}, & \text{if } p(\underline{0}) = 0, \\ 1\,\varepsilon_{\mathbb{N}_\infty}(x \mapsto p(1x)), & \text{otherwise,} \end{cases}$$

where the prefixing map $x \mapsto 1x \colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ plays the role of the successor function in $\mathbb{N}_\infty$, with ∞ as a fixed point. This amounts to a definition by co-recursion, using the fact that $\mathbb{N}_\infty$ is the final co-algebra of the functor $X \mapsto 1 + X$. Moreover, the definitions of ε_Y as in Theorem 7.5 and of ε_{Y_x} can also be naturally written in co-recursive form, using the fact that $2^{\mathbb{N}}$ is the final co-algebra of the functor $X \mapsto 2 \times X$. One is left wondering whether, for the manifestation of ordinals discussed here, that arise by iterating successor and squashed sums, starting from one, suitable co-induction and co-recursion principles generalizing those of $\mathbb{N}_\infty$ would be more natural than induction and recursion. Because our ordinals are retracts of $2^{\mathbb{N}}$, the co-induction principle for $2^{\mathbb{N}}$ is in a way inherited by them, but this is not the end of the story.

8 (Meta-mathematical) discussion

Coquand, Hancock and Setzer discuss [6] how one can transfinitetely iterate an $\mathbb{N}$-ary operation. For this purpose, they consider *limit structures*. A limit structure is a set X (which we will choose to be the Cantor space, or more precisely the space of selection functions $J2^{\mathbb{N}}$) together with an element $x \in X$ (think of the ordinal zero or one), a function $f \colon X \to X$ (think of the successor function), and $l \colon (\mathbb{N} \to X) \to X$ (they think of the supremum of a sequence of ordinals, but we think of the squashed sum). They show how any limit structure can reach γ-transfinite iteration for any $\gamma < \epsilon_0$ in the presence of simple types, and how to go beyond in richer type systems. Thus, with classical eyes, one can see how to get these ordinals embedded in the Cantor space with our constructions. An important question is how to see this with constructive eyes. In particular, we have accounted for transfinite induction but not (yet) transfinite iteration, and it remains to be seen how much ordinal arithmetic can be performed for ordinals in the Cantor space in the sense of Sections 5 and 6.

In the model of continuous functionals of HA^ω, searchable sets satisfy a compactness condition [7], and this is why ω cannot be embedded within HA^ω as an ordinal in our sense in the Cantor space. It is not hard to see that our embedding of ordinals always produces successor ordinals, or closed ordinal intervals $[0, \gamma]$, with the interval topology, which are compact as they should. The ordinals 2 and $\omega + 1$ can be embedded, but the ordinal $2^{\omega+1} = 2^\omega \times 2 = \omega \times 2$ cannot, because it is not a successor ordinal. Hence there is no hope of performing exponentiation for our notion of ordinal, although, as far as we know, nothing precludes the possibility of performing a construction that dominates exponentiation.

The argument of the first paragraph shows that it is not possible to embed ϵ_0 in $2^{\mathbb{N}}$ using limit structures. However, in principle it would be possible to embed $[0, \epsilon_0]$ and higher ordinals intervals in $2^{\mathbb{N}}$ within HA^ω without contradicting the fact that limit structures can only reach ordinals below ϵ_0. We don't dare to conjecture this or its negation, because we don't know enough about ordinals embedded in the Cantor space, in the sense of Sections 5 and 6, other than they are closed under successor and squashed sums, they are omniscient, they are retracts of the Cantor space, and have countable full subsets.

We conjecture that only sets with countable full subsets can be proved to be searchable in HA^ω. Assuming some of the contentious axioms discussed in the introduction, including continuity, it is well known that the Cantor space is searchable, but the Cantor space cannot be proved to be searchable in HA^ω [8]. Because, classically, any uncountable compact subset of the Cantor space has a copy of the Cantor space, it is plausible that a construction of a searchable set with no countable full subsets allows the construction of a selection function for the Cantor space. We leave this as an open problem, and this time we do dare to conjecture this.

The one-point compactification of the natural numbers $\mathbb{N}_\infty$ is the generic convergent sequence in the sense that any Cauchy sequence $\mathbb{N} \to X$ in a complete metric space X extends to a continuous map $\mathbb{N}_\infty \to X$ with ∞ mapped to the limit, and any continuous map $\mathbb{N}_\infty \to X$ arises in this way. The main results of [5] can be derived from our results and this observation.

The main (technically justified, philosophical) conclusion of this investigation is that it is misguided to assume that only the finite sets can satisfy the principle of omniscience in spartan constructive mathematics. We have shown that arbitrarily complex, infinite examples can be constructed, without assuming Brouwerian axioms (continuity, bar induction and fan theorem). As far as we know, this is new. The reason $\mathbb{N}$ fails to provably satisfy the principle of omniscience in all varieties of constructive mathematics is subtler and deeper than the mere fact that it is infinite. There must be many more, seemingly unlikely or counter-intuitive or surprising, instances of excluded middle to be found in spartan constructive mathematics.

9 Forerunners of the proof technique

I have found extremely difficult, even with the kind help of a number of colleagues in the research community, to give accurate historical references to the origin of the proof techniques and results developed here. Hence I report my own, somewhat accidental, involvement, supplemented by references provided by colleagues.

The work reported here has its origin in Exercise 1 of [1, page 581], which is attributed to Kreisel, although discussions with colleagues who were around at that time indicate that Kreisel may have learned this from Grilliot (see below). Working on programming language semantics, my experience with continuity led me to suspect that the classical set-theoretical model of Gödel's system T is not fully abstract. I asked a number of people in 2006, and Gordon Plotkin let me know that he had been shown a counter-example in the late 1970's, but that he couldn't remember it, and that a keyword to look for was *Kreisel*. When I mentioned this to Alex Simpson in mid 2007, he pointed me to the above exercise. I immediately realized that there was a connection with my work on searchable sets [11, 7], and wrote a little note [12], whose development is classical, and which eventually suffered a metamorphosis, resulting in the constructive work reported in this draft.

I discussed these results with a number of people in 2010, before this draft was written, sometimes from the point of view of classical higher-type computability theory with system T, and sometimes from the point of view of constructive mathematics, and Dag Normann saw a connection with "Grilliot's trick" [13, 14] in the context of higher-type computability with classical reasoning. The constructive development and proof techniques employed are more intimately connected to the closely related "Ishihara's trick" [15], particularly in its manifestations developed in [5], and I should have noticed the connection earlier because I was not unfamiliar with that. But fortunately I noticed the connection before Hajime Ishihara saw the first draft, and I warmly thank him for further bibliographic remarks and interesting technical discussions (not all reported here).

Thierry Coquand pointed out that Brouwer's 1927 proof of Theorem 1 of [19, page 459] can be seen as a forerunner of the trick, and Bridges mentioned Bishop's 1967 proof of Lemma 7 of [3, page 177]. John Longley mentioned the original proof of the Kreisel–Lacombe–Shoenfield theorem (in the Amsterdam proceedings volume of 1959), although the idea is perhaps somewhat obscured there by several other ingredients. This is compatible with the clean view of this theorem offered by Ishihara [15]. Longley also mentioned that he heard this sort of idea referred to as the *Kierstead phenomenon*, but he is unable to provide a reference.

In all cases, except the KLS theorem, the above authors build a Cauchy sequence from an infinite binary sequence in order to perform a constructive feat that amounts to a constructively valid instance of Markov's principle without assuming the principle, or, more ambitiously, a highly suspicious, but again constructively valid, instance of LPO. In the KLS theorem a sequence is constructed in a similar manner, but MP (or its weak version WMP [15]) is invoked in a step of the proof. All these constructions of Cauchy sequences from binary sequences amount to the fact that $\mathbb{N}_\infty$ is the generic convergent sequence in the sense discussed in Section 8.

References

[1] H.P. Barendregt. *The Lambda-Calculus: its Syntax and Semantics.* North-Holland, 1984.

[2] M.J. Beeson. *Foundations of Constructive Mathematics.* Springer, 1985.

[3] E. Bishop. *Foundations of constructive analysis.* McGraw-Hill Book Co., New York, 1967.

[4] D. Bridges and F. Richman. *Varieties of constructive mathematics*, volume 97 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1987.

[5] D. Bridges, D. van Dalen, and H. Ishihara. Ishihara's proof technique in constructive analysis. *Indag. Math. (N.S.)*, 14(2):163–168, 2003.

[6] T. Coquand, P. Hancock, and A. Setzer. Ordinals in type theory. Invited talk for Computer Science Logic (CSL), Aarhus, `http://www.cse.chalmers.se/~coquand/ordinal.ps`, August 1997.

[7] M. H. Escardó. Exhaustible sets in higher-type computation. *Log. Methods Comput. Sci.*, 4(3):3:3, 37, 2008.

[8] M. H. Escardó and P. Oliva. Bar recursion and products of selection functions. Available from the authors' web page, November 2010.

[9] M. H. Escardó and P. Oliva. Searchable sets, Dubuc–Penon compactness, omniscience principles, and the Drinker Paradox. In F. Ferreira, H. Guerra, E. Mayordomo, and J. Rasga, editors, *Computability in Europe 2010, Abstract and Handout Booklet*, pages 168–177. Centre fo Applied Mathematics and Information Technology, Department of Mathematics, University of Azores, 2010.

[10] M. H. Escardó and P. Oliva. Selection functions, bar recursion and backward induction. *Math. Structures Comput. Sci.*, 20, 2010.

[11] M.H. Escardó. Infinite sets that admit fast exhaustive search. In *Proceedings of the 22nd Annual IEEE Symposium on Logic In Computer Science*, pages 443–452. IEEE Computer Society, 2007.

[12] M.H. Escardó. Kreisel's counter-example to full abstraction of the set-theoretical model of Gödel's system T. Unpublished manuscript available at the author's web page, University of Birmingham, July 2007.

[13] T. J. Grilliot. On effectively discontinuous type-2 objects. *J. Symbolic Logic*, 36:245–248, 1971.

[14] J. P. Hartley. Effective discontinuity and a characterisation of the superjump. *J. Symbolic Logic*, 50(2):349–358, 1985.

[15] H. Ishihara. Continuity and nondiscontinuity in constructive mathematics. *J. Symbolic Logic*, 56(4):1349–1354, 1991.

[16] H. Ishihara. Constructive reverse mathematics: compactness properties. In *From sets and types to topology and analysis*, volume 48 of *Oxford Logic Guides*, pages 245–267. Oxford Univ. Press, Oxford, 2005.

[17] J. Lambek and P.J. Scott. *Introduction to Higher Order Categorical Logic.* Cambridge University Press, 1986.

[18] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics. An Introduction*, volume 121 and 123 of *Studies in Logic and the Foundations of Mathematics.* North Holland, Amsterdam, 1988.

[19] J. van Heijenoort. *From Frege to Gödel : A Source Book in Mathematical Logic, 1879-1931.* Harvard University Press, 1967.