Bilimits in the category of stably compact spaces and closed relations

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Contents

1 Introduction

Stably compact spaces have been studied in theoretical computer science for different reasons. One of them is that this spaces have a well-behaving order structure, which makes them interesting for denotational semantics. In fact many of the categories considered in domain theory can be embedded into the category of stably compact spaces. Another reason is that stably compact spaces are the correct generalization of compact Hausdorff spaces to T_0 spaces. These two faces of stably compact spaces make them worth considering, and virtually all results can be obtained either using topological or order-theoretical arguments. Many proofs throughout this paper will rely on the order structure of stably compact spaces, although the order is induced by the topology.

A *relation* between two sets A and B is nothing but a subset of the cartesian product $A \times B$. Depending on the type of relation one writes either $(a, b) \in R$ or $a\,R\,b$ to indicate that a pair $(a, b) \in A \times B$ is an element of the relation $R \subseteq A \times B$. A set P is called a *partially ordered set* or poset if there exists a subset $\sqsubseteq$ of $P \times P$ satisfying the following properties for all $x, y, z \in P$:

1. $x \sqsubseteq x$ (reflexive)
2. $(x \sqsubseteq y) \wedge (y \sqsubseteq x) \Rightarrow x = y$ (antisymmetric)
3. $(x \sqsubseteq y) \wedge (y \sqsubseteq z) \Rightarrow x \sqsubseteq z$ (transitive)

The smallest possible partial order is the equality relation, also called the flat order. We use the notation

$$\uparrow x = \{ y \in P \mid x \sqsubseteq y \}$$

for the *upper set* of an element and

$$\downarrow x = \{ y \in P \mid y \sqsubseteq x \}$$

for the *lower set* of x. For subsets $A \subseteq P$ we extend this to

$$\uparrow A = \bigcup_{a \in A} \uparrow a\,, \quad \downarrow A = \bigcup_{a \in A} \downarrow a\,.$$

Total orders are a special case of partial orders, but in general stably compact spaces will not bear a total order. A set $A \subseteq P$ is called *directed* if

$$\forall x, y \in A\, \exists z \in A : (x \sqsubseteq z) \wedge (y \sqsubseteq z).$$

The natural numbers $\mathbb{N}$ with the usual total order are a special example of a directed set. In fact directed index sets provide a generalization of sequences to nets, which are used to describe the topology in a space where points have an uncountable neighborhood basis. A partially ordered set P is called *directed complete* or dcpo if every directed subset D has a supremum, which is denoted by $\bigsqcup D$.

One can endow a topological space X with a relation $(\sqsubseteq_X) \subseteq (X \times X)$ by defining $x \sqsubseteq_X y$ if and only if x is in the closure of y. If X is a T_0 space then $\sqsubseteq_X$ is a partial order, the *specialization order*. Indeed, if $x \neq y \in X$ then by the T_0 separation axiom there exists either a neighborhood of x not containing y or a neighborhood of y not containing x. This proves the antisymmetry of the specialization order. Reflexivity is trivial, and transitivity follows from the topological properties of the closure. In general there are several topologies on X generating the same specialization order, the finest of which is the *Alexandroff topology* and the coarsest is known as the *upper topology*. One of the nice properties of stably compact spaces is that together with the specialization order the space is a dcpo. This is true for a bigger class of topological spaces, the sober spaces. A closed set $A \subseteq X$ of a topological space is *irreducible closed* or *union prime* if for every pair of closed sets B, C such that $A \subseteq B \cup C$ we have either $A \subseteq B$ or $A \subseteq C$. The space X is called *sober* if every irreducible closed set has a unique dense point, i.e. if every irreducible closed set is of the form $\downarrow x$ for some $x \in X$.

2 The category SCS^*

For a topological space $\mathcal{U}$ will denote the map which takes an element of the space and assigns its neighborhood filter to it, if the underlying topology is unambiguous. For a T_0 space X the specialization order can then be defined as

$$x \sqsubseteq_X x' \;:\Leftrightarrow\; \mathcal{U}(x) \subseteq \mathcal{U}(x')$$

A set is saturated if it is an upper set with respect to the specialization order. A space is stably compact if it is sober, compact, locally compact and finite intersections of compact saturated subsets are compact. Stably compact spaces have many interesting properties. For a brief description see [3]. One important fact is that the compact saturated subsets $\mathcal{K}(X)$ of a stably compact space X form a complete lattice, where suprema are intersections and finite infima are unions. This qualifies $\mathcal{K}(X)$ to serve as the closed sets of a new topology on X, the cocompact topology. The resulting topological space will be denoted by X_κ and the neighborhood filter of an element $x \in X$ will be referred to as $\mathcal{U}_\kappa(x)$ to distinguish it from the neighborhood filter in the original topology. These topologies are related in the following way:

Proposition 2.1 ([2], Prop. 2.4)**.** *Let X be a stably compact space.*

1. *The open sets of X_κ are the complements of compact saturated sets in X.*
2. *The open sets of X are complements of compact saturated sets in X_κ.*
3. *X_κ is stably compact and $(X_\kappa)_\kappa$ is identical to X.*
4. *The specialization order on X is the inverse of the specialization order on X_κ.*

The common refinement of the original and the cocompact topology on a stably compact space X is known as the patch topology. The set X together with its patch topology is denoted by X_π. Observe that X_π is always Hausdorff if X is stably compact, and $(X_\pi, \sqsubseteq_X)$ is a compact ordered space. We say that a function $f : X \to Y$ is patch continuous if $f : X_\pi \to Y_\pi$ is continuous.

2.1 Morphisms in SCS^*

We form the category SCS^* of stably compact spaces. For objects X and Y the set of morphisms $\mathsf{SCS}^*(X, Y)$ consists of relations $f \subseteq X \times Y$ such that f is closed in $X \times Y_\kappa$. For any relation $f \subseteq X \times Y$ we will denote its closure by $\overline{f}^{X \times Y_\kappa}$. We will write the composition of relations from left to right, unlike composition of functions. The specialization order on a stably compact space is the identity element: For all $f \in \mathsf{SCS}^*(X, Y)$ we have $(\sqsubseteq_X f \sqsubseteq_Y) = f$, which implies

$$(x, y) \in f \Rightarrow \downarrow x \times \uparrow y \subseteq f.$$

This reflects the fact that $\downarrow x = \overline{\{x\}}^X$ and $\uparrow y = \overline{\{y\}}^{Y_\kappa}$. The composition of morphisms is the usual relational composition: For $f \in \mathsf{SCS}^*(X, Y)$ and $g \in \mathsf{SCS}^*(Y, Z)$

$$(x, z) \in fg :\Leftrightarrow \exists y \in Y : (x, y) \in f \wedge (y, z) \in g.$$

For a relation $f \subseteq X \times Y$ we can form the transpose

$$(y, x) \in f^{\mathrm{T}} :\Leftrightarrow (x, y) \in f.$$

This is in general not a closed relation in our sense, but will prove to be a useful construction. Every compact Hausdorff space is stably compact, where the specialization order is the flat order. The graph of any continuous map $f : X \to Y$ between compact Hausdorff spaces is compact in $X \times Y = X \times Y_\kappa$. Moreover since $Y = Y_\kappa$ in the Hausdorff case, the category of compact Hausdorff spaces and closed relations embeds fully into SCS^*.

2.2 SCS^* as a cpo-enriched category

Observe that for stably compact spaces X and Y, the closed subsets of $X \times Y_\kappa$ can be ordered by reverse inclusion where $X \times Y$ is the minimal element. Obviously the set of morphisms $\mathsf{SCS}^*(X, Y)$ together with $\supseteq$ forms a cpo. We will denote this partial order on the set of morphisms by $\sqsubseteq$. This additional structure will be helpful when we consider limits and continuous endofunctors of SCS^*.

Definition 2.1 ([1], p. 322)**.** Let C be a category. We say C is a cpo-enriched category if

1. There exists a partial order $\sqsubseteq$ on every set of morphisms $\mathsf{C}(X, Y)$ such that $(\mathsf{C}(X, Y), \sqsubseteq)$ is a cpo.

2. For all objects X, Y, Z the composition of morphisms is continuous with respect to the Scott topologies on the product order $\mathsf{C}(X,Y) \times \mathsf{C}(Y,Z)$ and $\mathsf{C}(X,Z)$.

Lemma 2.2. SCS^* *is a cpo-enriched category.*

Proof. Let X, Y, Z be stably compact spaces. We already know that $\mathsf{SCS}^*(X,Y)$ and $\mathsf{SCS}^*(Y,Z)$ are cpos with reverse inclusion. The composition of relations is Scott-continuous if it preserves suprema of directed sets. Therefore it suffices to show: For all directed sets $R \subseteq \mathsf{SCS}^*(X,Y)$ and $S \subseteq \mathsf{SCS}^*(Y,Z)$, and for all $r_0 \in \mathsf{SCS}^*(X,Y)$, $s_0 \in \mathsf{SCS}^*(Y,Z)$

1. $$\left(\bigsqcup_{r\in R} r\right) s_0 = \bigsqcup_{r\in R} (r s_0)$$

2. $$r_0 \left(\bigsqcup_{s\in S} s\right) = \bigsqcup_{s\in S} (r_0 s).$$

The inclusion $\left(\bigsqcup_{r\in R} r\right) s_0 \subseteq \bigsqcup_{r\in R}(r s_0)$ is trivial. Fix $x \in X$ and assume $(x,z) \in \bigsqcup_{r\in R}(r s_0)$. The family

$$\big\{\{y \in Y \mid (x,y) \in r\} \,\big|\, r \in R\big\}$$

is a subset of $\mathcal{K}(Y)$, which are the closed sets in Y_κ. By assumption all sets in the family are not empty. Directedness of R yields the finite intersection property for the family. Hence by compactness of Y_κ the intersection

$$\left\{ y \in Y \,\middle|\, (x,y) \in \bigsqcup_{r\in R} r \right\}$$

is not empty. Therefore $(x,z) \in \left(\bigsqcup_{r\in R} r\right) s_0$ and so $\left(\bigsqcup_{r\in R} r\right) s_0 = \bigsqcup_{r\in R}(r s_0)$. To show the second equality $r_0 \left(\bigsqcup_{s\in S} s\right) = \bigsqcup_{s\in S}(r_0 s)$ we apply the first equality to the relations ${r_0}^{\mathrm{T}} \in \mathsf{SCS}^*(Y_\kappa, X_\kappa)$ and $\left\{s^{\mathrm{T}} \,\middle|\, s \in S\right\} \subseteq \mathsf{SCS}^*(Z_\kappa, Y_\kappa)$. □

Instead of working with the topological properties of a stably compact space one can consider the complete lattice of open sets $\Omega(X)$ together with subset inclusion. It turns out that by adjusting its action on relations one obtains a pair of contravariant functors $\Omega^* : \mathsf{SCS}^* \to \mathsf{SCF}^*$ and $\mathsf{pt}^* : \mathsf{SCF}^* \to \mathsf{SCS}^*$ between stably compact spaces and closed relations and the category of stably continuous frames and Scott continuous semilattice homomorphisms, which are the functors of a Stone duality. For details see [2], section 3. The functor pt^* takes a frame to the set of its completely prime filters.

3 Embedding projection pairs in SCS^*

The goal of this section is to reveal the structure of embedding projection pairs, referred to as *ep pairs*. In a slightly more general setting, as adjunctions, they have been studied in [2], and we will cite the relevant results.

3.1 ep pairs and surjective functions

Definition 3.1. Let X and Y be objects in the category SCS^*. An ep pair $f : X \to Y$ is a pair of morphisms $f^e \subseteq X \times Y$ and $f^p \subseteq Y \times X$ such that

$$f^e f^p = \mathrm{id}_X,\; f^p f^e \supseteq \mathrm{id}_Y\,.$$

Using the partial order on the sets $\mathsf{SCS}^*(X, X)$ and $\mathsf{SCS}^*(Y, Y)$ we can express the definition as

$$f^e f^p = \mathrm{id}_X,\; f^p f^e \sqsubseteq \mathrm{id}_Y\,.$$

Note that an adjunction between X and Y has the more general property

$$\mathrm{id}_X \sqsubseteq f^e f^p,\; f^p f^e \sqsubseteq \mathrm{id}_Y\,.$$

Here f^e is called the lower adjoint and f^p the upper adjoint. The class of objects in SCS^*together with ep pairs forms a new category, denoted by $(\mathsf{SCS}^*)^{\mathrm{ep}}$.

The most notable property of ep pairs is that the projection relation is always generated by a function, and the entire ep pair can be recovered using this function.

Lemma 3.1. *Let* $(e, p) : X \to Y$ *be an ep pair in* SCS^*. *Then*

$$\forall y \in Y\, \exists!\, x \in X : (y, x) \in p \wedge (x, y) \in e.$$

Proof. Existence of x follows from $pe \sqsubseteq \mathrm{id}_Y$. Suppose (y, x), $(y, x') \in p$ such that (x, y), $(x', y) \in e$. Then $(x, x') \in ep$ and $(x', x) \in ep$. Since ep is a partial order $x = x'$. □

We refer to this property of the projection relation as being *functional*. Corollary 4.8 in [2] now tells us that the corresponding function is *perfect*, meaning it is continuous with respect to the common refinement of the original and the cocompact topologies on domain and range. Moreover patch continuity is exactly the property that allows a function to give rise to an adjunction. The question how the function sits inside the relation is answered by Proposition 2.11 in [2].

Definition 3.2. Let $f : Y \to X$ be a function between T_0 spaces. Then the hypergraph of f is the set

$$\left\{(y, x) \in Y \times X \,\middle|\, f(y) \sqsubseteq_X x\right\}.$$

Similarly the left graph of f is the set

$$\left\{(x, y) \in X \times Y \,\middle|\, x \sqsubseteq_X f(y)\right\}.$$

Every upper adjoint is the hypergraph of its corresponding perfect function. More general, the hypergraph of any continuous function is a closed relation. Next we show the function of a projection relation is unique.

Lemma 3.2. *Let $(e,p) : X \to Y$ be an ep pair. Then p is the closure of a unique continuous function $\tilde{p} : Y \to X$.*

Proof. Let $\tilde{p}$ be the function sending y to the unique $x \in X$ such that $(y,x) \in p$ and $(x,y) \in e$. By Proposition 2.11 in [2] $\tilde{p}$ is continuous and p is the hypergraph

$$\bigcup_{y \in Y} \{y\} \times \uparrow \tilde{p}(y)\,.$$

Suppose $f : Y \to X$ is another continuous function such that $\overline{f}^{Y \times X_\kappa} = p$. Then

$$p = \bigcup_{y \in Y} \{y\} \times \uparrow f(y)\,.$$

Recall that the closure of a point in X_κ is its upper set with respect to $\sqsubseteq_X$. Therefore

$$\forall y \in Y : \uparrow \tilde{p}(y) \ = \uparrow f(y)$$

which by sobriety of X_κ (Proposition 2.4 in [2]) implies that $f = \tilde{p}$. □

We can also tell what the embedding part of an ep pair does. By Proposition 4.10 in [2] the lower adjoint is the left graph of the function which has the upper adjoint as its hypergraph. This yields the following characterization of ep pairs:

Lemma 3.3. *Let $(e,p) : X \to Y$ be an ep pair in SCS^*. Then p is the hypergraph of a perfect surjective function $\tilde{p} : Y \to X$ and e is the left graph of $\tilde{p}$. Conversely, if $\tilde{p} : Y \to X$ is a perfect surjective function then the left graph and the hypergraph of $\tilde{p}$ form an ep pair.*

Proof. If $(e,p) : X \to Y$ is an ep pair, then $ep = \mathrm{id}_X$, which implies

$$\forall x \in X\, \exists y \in Y : \underbrace{x \sqsubseteq_X \tilde{p}(y)}_{(x,y) \in e} \wedge \underbrace{\tilde{p}(y) \sqsubseteq_X x}_{(y,x) \in p}\,.$$

This implies surjectivity of $\tilde{p}$. Conversely if $\tilde{p}$ is a perfect surjection then the left graph and the hypergraph form an adjoint pair of morphisms. Let e be the left graph of $\tilde{p}$ and p be the hypergraph. It remains to show that $\mathrm{id}_X \subseteq ep$. For any pair (x,x') in id_X choose $y \in Y$ such that $\tilde{p}(y) = x$. Then $(x,y) \in e$ and since $\tilde{p}(y) = x \sqsubseteq_X x'$ we also have $(y,x') \in p$. Hence $ep = \mathrm{id}_X$ and (e,p) is an ep pair. □

Corollary. *If X is a T_1 space, then p is a surjective function.*

Proof. If X is T_1 then $X_\kappa = X$ and $\uparrow \tilde{p}(y) = \{\tilde{p}(y)\}$ for all $y \in Y$, so $p = \tilde{p}$. □

There is another characterization of the embedding part in an ep pair, described by the followig lemma.

Lemma 3.4. *Let $(e, p) : X \to Y$ be an ep pair. Then e is the closure of $\tilde{p}^{\mathrm{T}}$, where $\tilde{p}$ is the perfect function associated to p.*

Proof. Let $\tilde{e} = \tilde{p}^{\mathrm{T}}$. Since $\tilde{p} \subseteq p$ we have $\tilde{e} \subseteq e$ and by $\overline{e} = e$ we have $\overline{\tilde{e}} \subseteq e$. It remains to show the other inclusion. Let $(x, y) \in e$. Then $(x, \tilde{p}(y)) \in pe$, so $x \sqsubseteq_X \tilde{p}(y)$, which means $x \in \downarrow \tilde{p}(y)$. Recall that $\downarrow \tilde{p}(y) = \overline{\{\tilde{p}(y)\}}^X$ whence $x \in \overline{\{\tilde{p}(y)\}}^X$. This implies $(x, y) \in \overline{\{(\tilde{p}(y), y)\}}^{X \times Y_\kappa}$. □

Corollary. *If X is T_1, then any ep pair consists of a continuous surjective function and its transpose.*

3.2 Distributive laws

This subsection provides two useful results, which state that closing a relation given by a perfect function distributes over composition.

Lemma 3.5. *Let $f : X \to Y$, $g : Y \to Z$ be continuous functions between stably compact spaces. Then*

$$\overline{g \circ f}^{X \times Z_\kappa} = \left(\overline{f}^{X \times Y_\kappa}\right)\left(\overline{g}^{Y \times Z_\kappa}\right).$$

Proof. The closures of the functions are their hypergraphs. For simplicity we write

$$\overline{f} = \overline{f}^{X \times Y_\kappa}, \, \overline{g} = \overline{g}^{Y \times Z_\kappa}.$$

By definition we have

$$(x, z) \in \overline{f}\overline{g} \Leftrightarrow \exists y \in \uparrow f(x) \, : \, z \in \uparrow g(y) \, .$$

Recall that continuous functions are monotone with respect to the specialization orders. Therefore

$$f(x) \sqsubseteq_Y y \Rightarrow (g \circ f)(x) \sqsubseteq_Z g(y) \sqsubseteq_Z z$$

whence $(x, z) \in \overline{g \circ f}$. Now suppose $(x, z) \in \overline{g \circ f}$. This is the case if and only if $(g \circ f)(x) \sqsubseteq_Z z$. Let $y := f(x)$. Certainly $f(x) \sqsubseteq_Y y$ and $(g \circ f)(x) = g(y) \sqsubseteq_Z z$. Hence $\overline{g \circ f} = \overline{f}\overline{g}$. □

Unfortunately this distributive law does not hold true for the transposes of arbitrary continuous functions. In the case of perfect functions we do have the distributivity:

Lemma 3.6. *Let $f : X \to Y$, $g : Y \to Z$ be perfect functions between stably compact spaces. Then*

$$\overline{(g \circ f)^{\mathrm{T}}}^{Z \times X_\kappa} = \overline{g^{\mathrm{T}}}^{Z \times Y_\kappa} \, \overline{f^{\mathrm{T}}}^{Y \times X_\kappa}.$$

Proof. The closures of the transposes are their left graphs. Therefore

$$\begin{aligned}(z,x) \in \overline{(g \circ f)^{\mathrm{T}}} \quad &\Leftrightarrow \quad z \sqsubseteq_Z (g \circ f)(x), \\ (z,x) \in \overline{g^{\mathrm{T}}}\, \overline{f^{\mathrm{T}}} \quad &\Leftrightarrow \quad \exists y \in Y : z \sqsubseteq_Z g(y) \wedge y \sqsubseteq_Y f(x).\end{aligned}$$

Suppose $(z,x) \in \overline{g^{\mathrm{T}}}\, \overline{f^{\mathrm{T}}}$ and let y as above. By continuity of g we have $g(y) \sqsubseteq_Z (g \circ f)(x)$ and so

$$z \sqsubseteq_Z g(y) \sqsubseteq_Z (g \circ f)(x)$$

whence $(z,x) \in \overline{(g \circ f)^{\mathrm{T}}}$.

Now suppose $(z,x) \in \overline{(g \circ f)^{\mathrm{T}}}$. Let $y = f(x)$. Then $y \sqsubseteq_Y f(x)$ and $z \sqsubseteq_Z g(y) = (g \circ f)(x)$. Hence $(z,x) \in \overline{g^{\mathrm{T}}}\, \overline{f^{\mathrm{T}}}$. □

4 Colimits of directed diagrams in $(\mathsf{SCS}^*)^{\mathrm{ep}}$

In the following let $(A, \sqsubseteq)$ be a directed set. A can be considered as a category where there is a morphism $\alpha \to \beta$ if and only if $\alpha \sqsubseteq \beta$. A directed diagram in $(\mathsf{SCS}^*)^{\mathrm{ep}}$ is a functor Δ from A to $(\mathsf{SCS}^*)^{\mathrm{ep}}$, that is a set of objects $\{X_\alpha\}_{\alpha \in A}$ together with ep pairs $f_{\alpha\beta} : X_\alpha \to X_\beta = \Delta(\alpha \sqsubseteq \beta)$ whenever $\alpha \sqsubseteq \beta$ such that

$$\forall \alpha,\, \beta,\, \gamma \in A,\, \alpha \sqsubseteq \beta \sqsubseteq \gamma : \; f_{\alpha\beta}\, f_{\beta\gamma} = f_{\alpha\gamma}.$$

In particular $f_{\beta\beta} = \mathrm{id}_{X_\beta}$ for all $\beta \in A$. In this section we shall construct a colimit for such a directed diagram. Note that the category SCS of stably compact spaces has arbitrary products (Proposition 1.3.4 in [3]). This is why the topological product $\prod_{\alpha \in A} X_\alpha$ is a stably compact space.

4.1 Δ-closed and Δ-invariant sets

Let $\Delta : A \to (\mathsf{SCS}^*)^{\mathrm{ep}}$ be a functor. Then for $\alpha \sqsubseteq \beta \in A$, $\Delta(\alpha \sqsubseteq \beta)$ is a morphism from $\Delta(\alpha)$ to $\Delta(\beta)$. Since $\Delta(\alpha \sqsubseteq \beta)$ is an adjunction, there exists a unique perfect surjection $p_{\alpha\beta} : X_\beta \to X_\alpha$ associated to it. Therefore we can think of Δ as a contravariant functor from A into the category of stably compact spaces and perfect surjections. Since the hypergraph of a perfect surjection is an upper adjoint we call the category of stably compact spaces with continuous surjections SCS^*_u.

Definition 4.1. Let A be a directed set and $\Delta : A \to \mathsf{SCS}^*_u$ be a functor. Let $p_{\alpha\beta} = \Delta(\alpha \sqsubseteq \beta)$ denote the morphisms. For all $\beta \in A$ define

$$i_\beta : \prod_{\alpha \in A} \Delta(\alpha) \to \prod_{\alpha \in A} \Delta(\alpha), \quad i_\beta((x_\alpha)_{\alpha \in A})_\gamma = \begin{cases} p_{\gamma\beta}(x_\beta) & \text{if } \gamma \in {\downarrow}\beta \\ x_\gamma & \text{otherwise} \end{cases}.$$

Then a subset Y of $\prod_{\alpha \in A} \Delta(\alpha)$ is called Δ-closed if Y is closed in the patch topology $\prod_{\alpha \in A} \Delta(\alpha)_\kappa$ and

$$\forall \beta \in A : \; i_\beta(Y) \subseteq Y.$$

We call Y a Δ-invariant set if additionally

$$\forall \beta \in A : i_\beta(Y) = Y.$$

Of course the empty set is Δ-invariant. We define the biggest pointwise Δ-invariant set

$$X = \left\{ (x_\alpha)_{\alpha \in A} \,\middle|\, \forall \beta \in A : i_\beta((x_\alpha)_{\alpha \in A}) = (x_\alpha)_{\alpha \in A} \right\}. \tag{4.1}$$

We will see that X is not empty and X is the colimit the diagram given by Δ.

In the following we will let X_α denote the object $\Delta(\alpha)$, to make the notation more convenient.

Lemma 4.1. *For $\beta \in A$ let $p_\beta = \pi_\beta\big|_X$ be the restriction of the canonical projection $\prod_{\alpha \in A} X_\alpha \to X_\beta$. Then p_β is a perfect map.*

Proof. It suffices to show that π_β is a perfect map. Obviously π_β is continuous. To obtain patch continuity and thereby perfectness we show that preimages of compact saturated subsets of X_β are compact and saturated, because these are the closed sets with respect to the cocompact topologies. Observe that the specialization order on the product space $\prod_{\alpha \in A} X_\alpha$ is the product order, meaning a set is upper if and only if each projection onto a component x_α is upper. Let $K \subseteq X_\beta$ be compact and saturated. Then

$$\pi_\beta^{-1}(K) = K \times \prod_{\alpha \neq \beta} X_\alpha$$

which is clearly an upper set. Since all the X_α are compact $\pi_\beta^{-1}(K)$ is compact. □

Corollary 4.2. *Let i_β be defined as in Definition 4.1 and let*

$$\pi_{\downarrow\beta} : \prod_{\alpha \in A} X_\alpha \to \prod_{\alpha \in \downarrow\beta} X_\alpha \tag{4.2}$$

be the canonical projection. Then i_β and $\pi_{\downarrow\beta}$ are perfect functions.

Proof. The composition of either i_β or $\pi_{\downarrow\beta}$ with any applicable canonical single projection π_γ is either π_γ or $\pi_\gamma \circ p_{\gamma\beta}$, both of which are patch continuous. Hence i_β and $\pi_{\downarrow\beta}$ are patch continuous. □

Next we will prove a useful fact about Δ-closed sets.

Theorem 4.3. *Let Δ and $\{p_{\alpha\beta}\}_{\alpha \sqsubseteq \beta \in A}$ be defined as in Definition 4.1. Then every nonempty Δ-closed set intersects the maximal pointwise Δ-invariant set X.*

Proof. The statement of the theorem implies that the maximal pointwise Δ-invariant set X is not empty. Let Y be a nonempty Δ-closed set. Since $\prod_{\alpha \in A} (X_\alpha)_\kappa$ is compact and Hausdorff, closed is equivalent to compact. Y

is closed, therefore compact. The image $i_\alpha(Y)$ is compact for all $\alpha \in A$ and thereby closed. Let $B \subset A$ be finite. By directedness of A there exists $\beta \in A$ such that $B \subseteq \downarrow\beta$. Observe that

$$\alpha \sqsubseteq \beta \Rightarrow i_\beta(Y) \subseteq i_\alpha(Y).$$

Hence

$$\emptyset \neq i_\beta(Y) \subseteq \bigcap_{\alpha\in B} i_\alpha(Y)$$

so the family $\{i_\alpha(Y)\}_{\alpha\in A}$ has the finite intersection property. By compactness the intersection of the entire family is not empty. We claim that

$$\tilde{Y} = \bigcap_{\alpha\in A} i_\alpha(Y) \subseteq Y \cap X.$$

Let τ, $\epsilon \in A$ such that $\tau \sqsubseteq \epsilon$. For all $\alpha \in \uparrow\epsilon$ and all $y \in Y$ we have

$$(i_\alpha(y))_\tau = p_{\tau\epsilon}((i_\alpha(y))_\epsilon)$$

so all $y \in \tilde{Y}$ satisfy

$$y_\tau = p_{\tau\epsilon}(y_\epsilon).$$

Since τ and ϵ were arbitrary we have $\tilde{Y} \subseteq X$. □

4.2 The maximal pointwise Δ-invariant set as the colimit

Lemma 4.4. *Let $\Delta = (X_\alpha, f_{\alpha\beta})_{\alpha\sqsubseteq\beta\in A}$ be a directed diagram in $(\mathsf{SCS}^*)^{ep}$ where for all $\alpha \sqsubseteq \beta$ the relation $f^p_{\alpha\beta}$ is the hypergraph of a perfect surjective function $p_{\alpha\beta} : X_\beta \to X_\alpha$. Fix $\beta \in A$ and let $\pi_{\downarrow\beta}$ be the canonical projection and X the maximal pointwise Δ-invariant set. Then $\pi_{\downarrow\beta}(X)$ is compact in $\prod_{\alpha\in\downarrow\beta} X_\alpha$.*

Proof. Define $f : X_\beta \to \prod_{\alpha\in\downarrow\beta} X_\alpha$ by

$$f(x) = (p_{\alpha\beta}(x))_{\alpha\in\downarrow\beta}.$$

This function is continuous and since $p_{\beta\beta}$ is the identity function, f can be regarded as its own graph. Since all of the spaces X_α are compact and the function f is continuous, the image of X_β

$$f(X_\beta) = \left\{(x_\alpha)_{\alpha\in\downarrow\beta} \,\middle|\, \exists x \in X_\beta : \forall\alpha \in\downarrow\beta : x_\alpha = p_{\alpha\beta}(x)\right\}$$

is compact. Now observe that $f(X_\beta)$ is actually equal to $\pi_{\downarrow\beta}(X)$, whence $\pi_{\downarrow\beta}(X)$ is compact. □

With this we are able to show that X is compact.

Theorem 4.5. *Let X be defined as in (4.1). Then X is compact.*

Proof. As in the proof of Lemma 4.4 we define

$$f_\beta : X_\beta \to \prod_{\alpha \in \Downarrow\beta} X_\alpha, \quad x \mapsto (p_{\alpha\beta}(x))_{\alpha \in \Downarrow\beta}$$

for all $\beta \in A$. By Lemma 4.4 $f_\beta(X_\beta)$ is compact. Form the saturation:

$$\uparrow f_\beta(X_\beta) = \left\{(x_\alpha)_{\alpha\in\Downarrow\beta} \,\middle|\, \exists x \in X_\beta : \forall \alpha \in\downarrow\beta : p_{\alpha\beta}(x) \sqsubseteq_{X_\alpha} x_\alpha\right\}.$$

Let $\pi_{\Downarrow\beta}$ be the map defined in (4.2). Then

$$\pi^{-1}_{\Downarrow\beta}\left(\uparrow f_\beta(X_\beta)\right) = \left\{(x_\alpha)_{\alpha\in A} \,\middle|\, \exists x \in X_\beta : \forall \alpha \in\downarrow\beta : p_{\alpha\beta}(x) \sqsubseteq_{X_\alpha} x_\alpha\right\}.$$

Recall that $\pi_{\Downarrow\beta}$ is a perfect map, whence $\pi^{-1}_{\Downarrow\beta}\left(\uparrow f_\beta(X_\beta)\right)$ is again compact and saturated. Since $\prod_{\alpha\in A} X_\alpha$ is a stably compact space the set of all compact saturated subsets with $\supseteq$ is a complete lattice. We may form the supremum

$$\tilde{X} = \bigsqcup_{\beta\in A} \pi^{-1}_{\Downarrow\beta}\left(\uparrow f_\beta(X_\beta)\right)$$

which is still a compact saturated set. Concretely

$$\tilde{X} = \left\{(x_\alpha)_{\alpha\in A} \,\middle|\, \forall \beta \in A\, \exists x \in X_\beta : \forall \alpha \in\downarrow\beta : p_{\alpha\beta}(x) \sqsubseteq_{X_\alpha} x_\alpha\right\}.$$

Obviously $X \subseteq\uparrow X \subseteq \tilde{X}$. We show that $\tilde{X} \subseteq\uparrow X$. Fix $(x_\alpha)_{\alpha\in A} \in \tilde{X}$ and define

$$Y = \left\{(y_\beta)_{\beta\in A} \,\middle|\, \forall \beta \in A, \forall \alpha \in\downarrow\beta : p_{\alpha\beta}(y_\beta) \sqsubseteq_{X_\alpha} x_\alpha\right\}.$$

By definition Y is not empty. We can rewrite

$$Y = \bigcap_{\beta\in A} \pi^{-1}_\beta\left(\bigcap_{\alpha\in\Downarrow\beta} p^{-1}_{\alpha\beta}(\downarrow x_\alpha)\right)$$

and since $\downarrow x_\alpha$ is patch closed for all $\alpha \in A$ and all involved functions are patch continuous, Y is patch closed. Observe that $i_\alpha(Y) \subseteq Y$ for all $\alpha \in A$: If $(y_\beta)_{\beta\in A} \in Y$ and $\gamma \in A$, then for a fixed $\gamma \in A$ we have $\forall \alpha \in\downarrow\gamma : p_{\alpha\gamma}(y_\gamma) \sqsubseteq_{X_\alpha} x_\alpha$, whence $i_\gamma((y_\beta)_{\beta\in A})$ still satisfies the defining property of Y. Hence Y is Δ-closed. By Theorem 4.3 its intersection with X is not empty. Therefore we have $(x'_\alpha)_{\alpha\in A} \in X$ such that $(x_\alpha)_{\alpha\in A} \in\uparrow (x'_\alpha)_{\alpha\in A}$. Since $(x_\alpha)_{\alpha\in A} \in \tilde{X}$ was arbitrary we have $\tilde{X} \subseteq\uparrow X$ and thereby compactness of $\uparrow X$ and X. □

Since X is a compact subset of a stably compact space, X itself is an object in SCS^*. There is an elegant way of proving that X is colimiting our directed diagram. Recall that SCS^* is a cpo-enriched category. By theorem 10.4 in [1] it suffices to find a cocone $\mu : \Delta \to X$ such that

$$\bigsqcup_{\alpha\in A} \mu^p_\alpha\, \mu^e_\alpha = \mathrm{id}_X\,.$$

The ep pairs $\mu_\alpha : X_\alpha \to X$ can be defined in the obvious way: We generate them by the canonical projections onto single components. By now we know that p_β gives rise to an adjunction between X_β and X. For this adjunction to become an ep pair we need surjectivity of p_β.

Proposition 4.6. *Let p_β be defined as in Lemma 4.1. Then p_β is surjective.*

Proof. Fix $x \in X_\beta$. We have to show that there exists an element $(y_\alpha)_{\alpha\in A} \in X$ such that $y_\beta = x$. Let

$$Y = \left(\prod_{\alpha \in \Uparrow\beta} p_{\beta\alpha}^{-1}(\{x\})\right) \times \left(\prod_{\alpha \notin \Uparrow\beta} X_\alpha\right).$$

Observe that since all $p_{\beta\alpha}$ are surjective this set is not empty. Further $\{x\}$ is closed in $(X_\beta)_\kappa$ and all $p_{\beta\alpha}$ are patch continuous. Hence Y is patch closed. Suppose $\gamma \in\uparrow\beta$ and $\beta \sqsubseteq \delta \sqsubseteq \gamma$. Then if $\mathrm{y} = (y_\alpha)_{\alpha\in A} \in Y$

$$p_{\beta\delta} \circ \underbrace{p_{\delta\gamma}(y_\gamma)}_{=(i_\gamma(\mathrm{y}))_\delta} = p_{\beta\gamma}(y_\gamma) = x.$$

This shows that Y is Δ-closed and thereby has nonempty intersection with X. □

We now may form an ep pair $\mu_\beta : X_\beta \to X$ by taking μ_β^e to be the left graph of p_β and μ_β^p to be the hypergraph of p_β. Observe that by construction of X we have $p_\beta = p_{\beta\gamma} \circ p_\gamma$ for all $\beta \sqsubseteq \gamma \in A$. By Lemma 3.5 and Lemma 3.6 we have $\mu_\beta^p = \mu_\gamma^p f_{\beta\gamma}^p$ and $\mu_\beta^e = f_{\beta\gamma}^e \mu_\gamma^e$ for all $\beta \sqsubseteq \gamma \in A$. This makes $\mu : \Delta \to X$ into a cocone over Δ. Further note that for any $\beta \in A$

$$((x_\alpha)_{\alpha\in A}, (y_\alpha)_{\alpha\in A}) \in \mu_\beta^p \mu_\beta^e \Leftrightarrow x_\beta \sqsubseteq_{X_\beta} y_\beta$$

whence

$$\bigsqcup_{\beta\in A} \mu_\beta^p \mu_\beta^e = \bigcap_{\beta\in A} \mu_\beta^p \mu_\beta^e = \sqsubseteq_X = \mathrm{id}_X .$$

Thus X is the colimit for the directed diagram Δ in $(\mathsf{SCS}^*)^{\mathrm{ep}}$. Further $\mu^e : \Delta^e \to X$ is a colimiting cocone for the diagram

$$\Delta^e = (X_\alpha,\, f_{\alpha\beta}^e)_{\alpha\sqsubseteq\beta\in A}$$

in SCS^* and $\mu^p : X \to \Delta^p$ is a limiting cone for the covariant diagram

$$\Delta^p = (X_\alpha,\, f_{\alpha\beta}^p)_{\alpha\sqsubseteq\beta\in A}.$$

We have proven the following theorem:

Theorem 4.7. *Let SCS_l^* denote the category of stably compact spaces and lower adjoints. Every directed diagram in SCS_l^* has a bilimit.*

This result was already established in [2], where it was proven by using the Stone duality between SCS^* and the category of stably continuous frames SCF^*. Since the elements of an adjuction uniquely determine each other, it does not make a difference whether one considers SCS_l^*, SCS_u^* or $(\mathsf{SCS}^*)^{\mathrm{ep}}$. Now we have a purely topological construction for the bilimit.

5 Fixed points of endofunctors

In this section we want to apply our results to show that fixed points of certain endofunctors exist. Once more the fact that SCS^* is a cpo-enriched category will simplify our proofs. A useful property of a category is having locally determined colimits:

Definition 5.1 ([1], p. 325)**.** A cpo-enriched category C has locally determined colimits if for every directed diagram $\Delta : I \to \mathsf{C}^{\mathrm{ep}}$ and every cocone $\mu : \Delta \to X$ the following are equivalent:

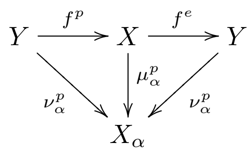
(i) μ is colimiting.

(ii) $\bigsqcup_{i \in I} \mu_i^e \circ \mu_i^p = \mathrm{id}_X$.

We already know that the implication (ii) $\Rightarrow$ (i) is always true. For SCS^* we show the other implication.

Lemma 5.1. *Let $\Delta = (X_\alpha, f_{\alpha\beta})_{\alpha \sqsubseteq \beta \in A}$ be a directed diagram in $(\mathsf{SCS}^*)^{\mathrm{ep}}$ and let $\mu : \Delta \to X$ and $\nu : \Delta \to Y$ be colimiting cocones. Let $f : X \to Y$ be the mediating morphism. Then $f^p f^e = \mathrm{id}_Y$.*

Proof. Choose $\alpha \in A$. Since f is mediating, the following diagram commutes:



which implies $f^p f^e \nu_\alpha^p = \nu_\alpha^p$. On the other hand $\mathrm{id}_Y \, \nu_\alpha^p = \nu_\alpha^p$. Since the mediating arrow is unique we obtain $f^p f^e = \mathrm{id}_Y$. □

Corollary 5.2. *Any two colimits of a directed diagram $\Delta : I \to (\mathsf{SCS}^*)^{\mathrm{ep}}$ are homeomorphic.*

Proof. Let $\mu : \Delta \to X$ and $\nu : \Delta \to Y$ be colimiting cocones and $f : X \to Y$ be the mediating arrow. Then by Lemma 5.1

$$f^e f^p = \mathrm{id}_X \text{ and } f^p f^e = \mathrm{id}_Y .$$

Therefore both (f^e, f^p) and (f^p, f^e) are ep pairs, whence there exist perfect surjections $e : X \to Y$ and $p : Y \to X$ such that $f^e = \overline{e}$ and $f^p = \overline{p}$. Let i_X and i_Y denote the identity functions on X and Y, respectively. We have

$$\overline{p \circ e} = \overline{i_X} = \mathrm{id}_X \text{ and } \overline{e \circ p} = \overline{i_Y} = \mathrm{id}_Y .$$

For a functional relation there is only one function such that the relation is its closure. Hence

$$p \circ e = i_X \text{ and } e \circ p = i_Y .$$

□

We use Lemma 5.1 to prove that SCS^* has locally determined colimits.

Lemma 5.3. SCS^* *has locally determined colimits.*

Proof. For a diagram $\Delta : A \to (\mathsf{SCS}^*)^{\mathrm{ep}}$ let $\mu : \Delta \to X$ be the colimiting cocone as constructed in (4.1). Let $\nu : \Delta \to Y$ be another colimiting cocone, and let $f : X \to Y$ be the mediating morphism in $(\mathsf{SCS}^*)^{\mathrm{ep}}$. Then

$$\bigsqcup_{\alpha \in A} \nu^p_\alpha \nu^e_\alpha = \bigsqcup_{\alpha \in A} f^p \mu^p_\alpha \mu^e_\alpha f^e$$

and by Scott-continuity of composition this equals

$$f^p \left(\bigsqcup_{\alpha \in A} \mu^p_\alpha \mu^e_\alpha \right) f^e = f^p f^e \overset{\text{Lemma 5.1}}{=} \mathrm{id}_Y \,.$$

□

Definition 5.2. Let C and D be cpo-enriched categories and F be a functor between C and D. Then F is said to be locally continuous if

$$f \sqsubseteq g \Rightarrow F(f) \sqsubseteq F(g)$$

for all morphisms with the same domain and codomain and F preserves directed suprema in the hom-sets:

$$F\left(\bigsqcup_{d \in D} d \right) = \bigsqcup_{d \in D} F(d).$$

A functor $G : \mathsf{C}^{\mathrm{ep}} \to \mathsf{D}^{\mathrm{ep}}$ is said to be locally continuous if the components F^e and F^p are locally continuous.

Definition 5.3. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between cpo-enriched categories. Then F is said to be continuous if it preserves colimits of directed diagrams.

Note that a functor $F : \mathsf{C} \to \mathsf{D}$ induces a functor $F^{\mathrm{ep}} : \mathsf{C}^{\mathrm{ep}} \to \mathsf{D}^{\mathrm{ep}}$ by

$$F^{\mathrm{ep}} X = FX, \quad F^{\mathrm{ep}} \left(X \underset{f_p}{\overset{f^e}{\rightleftarrows}} Y \right) = FX \underset{F(f_p)}{\overset{F(f^e)}{\rightleftarrows}} FY \,.$$

Local continuity is a property which is relatively easy to check. For the induced functor between the categories with ep pairs even more holds:

Lemma 5.4 ([1], Lemma 10.7)**.** *If F is a locally continuous functor between cpo-enriched categories C and D, then the induced functor between C^{ep} and D^{ep} is continuous.*

Observe that in SCS^* the empty set is the initial object, because for every stably compact space X the space $\emptyset \times X_\kappa$ is the empty set and thus allows only one relation, the empty relation. In $(\mathsf{SCS}^*)^{\mathrm{ep}}$ the situation is different:

Lemma 5.5. *The one-point space is the initial object in* $(\mathsf{SCS}^*)^{\mathrm{ep}}$.

Proof. Let $\mathbf{1} = \{*\}$ denote the one-point set. Let X be an object in $(\mathsf{SCS}^*)^{\mathrm{ep}}$ and (e, p) be an ep pair $\mathbf{1} \to X$. Since $pe \sqsubseteq \mathrm{id}_X$, pe contains the diagonal. Therefore

$$\forall x \in X\, \exists y \in \mathbf{1} : \ (x, y) \in p \wedge (y, x) \in e.$$

But y is unique, and so

$$\begin{aligned} X \times \mathbf{1} &= \bigcup_{x \in X} (x, *) \subseteq p \\ \mathbf{1} \times X &= \bigcup_{x \in X} (*, x) \subseteq e \end{aligned}$$

which implies $p = X \times \mathbf{1}$ and $e = \mathbf{1} \times X$. □

This puts us in the lucky situation that fixed points for endfunctors in SCS^* can be nontrivial. For a locally continuous endofunctor $F : \mathsf{SCS}^* \to \mathsf{SCS}^*$ consider the unique arrow

$$f : \mathbf{1} \to F(\mathbf{1})$$

in the category $(\mathsf{SCS}^*)^{\mathrm{ep}}$. We form a directed diagram $\Delta : \mathbb{N} \to (\mathsf{SCS}^*)^{\mathrm{ep}}$ by

$$\begin{aligned} \Delta(n) &= F^n(\mathbf{1}) \\ \Delta(n \leq n+1) &= F^n(f) : F^n(\mathbf{1}) \to F^{n+1}(\mathbf{1}). \end{aligned}$$

The colimit of this diagram will be a fixed point for F, as shown in the following Proposition.

Proposition 5.6. *Let F be a locally continuous endofunctor in SCS^*. Let X be a stably compact space and suppose there exists a morphism $f \in (\mathsf{SCS}^*)^{\mathrm{ep}}(X, FX)$, where F in this case is the induced continuous endofunctor in $(\mathsf{SCS}^*)^{\mathrm{ep}}$. Then there exists a fixed point Y for the functor, that is a stably compact space such that*

$$Y \cong F(Y)$$

and an ep pair $X \to Y$.

Proof. We form a diagram $\Delta : \mathbb{N} \to (\mathsf{SCS}^*)^{\mathrm{ep}}$ by

$$\begin{aligned} \Delta(n) &= F^n(X) \\ \Delta(n \leq n+1) &= F^n(f) : F^n(X) \to F^{n+1}(X). \end{aligned}$$

Apply F to this diagram to obtain

$$\begin{aligned} F(\Delta)(n) &= F^{n+1}(X) \\ F(\Delta)(n \leq n+1) &= F^{n+1}(f) : F^{n+1}(X) \to F^{n+2}(X). \end{aligned}$$

By theorem 4.7 there exists a colimit for Δ, which is

$$Y = \left\{ (x_n)_{n\geq 0} \in \prod_{n\geq 0} F^n(X) \,\middle|\, \forall n \in \mathbb{N} : x_n = F^n(\tilde{f})(x_{n+1}) \right\}$$

where $\tilde{f} : FX \to X$ is the perfect surjection associated to f^p and $F^n(\tilde{f})$ the perfect surjection associated to $(F^n(f))^p$. By continuity of F we know that FY is colimiting the diagram $F(\Delta)$, whence FY is of the form

$$FY = \left\{ (x_n)_{n\geq 1} \in \prod_{n\geq 1} F^n(X) \,\middle|\, \forall n \geq 1 : x_n = F^n(\tilde{f})(x_{n+1}) \right\}.$$

The colimiting cocone $\mu : \Delta \to Y$ is generated by the projections

$$p_j : Y \to F^j X, \quad (x_n)_{n\geq 0} \mapsto x_j.$$

Let similarly $q_j : FY \to F^j X$ denote the projections generating the colimiting cocone $\nu : F(\Delta) \to FY$. Define

$$\varphi : Y \to FY, \quad (x_n)_{n\geq 0} \mapsto (x_n)_{n\geq 1}.$$

This is obviously a patch continuous surjection, since for all $j \geq 1$ we have $q_j \circ \varphi = p_j$. There is an inverse for φ: For $(x_n)_{n\geq 1} \in FY$ let

$$\psi((x_n)_{n\geq 1}) = (\tilde{f}(x_{n+1}))_{n\geq 0}.$$

Here we have $p_j \circ \psi = F^j(\tilde{f}) \circ q_{j+1}$ for all $j \geq 0$, so ψ is patch continuous and surjective as well. Obviously ψ and φ are mutually inverse. Hence $Y \cong FY$. Further (μ_0^e, μ_0^p) is an ep pair between X and Y. □

5.1 The Cantor set as a fixed point

In the following we will show that the Cantor set is homeomorphic to the disjoint union with itself.

5.1.1 The coproduct functor

In a category where finite coproducts exist, the assignment $F : X \mapsto X \oplus X$ is always functorial. In SCS^* the coproduct is given by disjoint union [1]. If $f : X \to Y$ is a morphism, then $F(f)$ is just the disjoint union of f with itself, where f is regarded as a subset of $X \times Y$. More precisely, write $X \oplus X = \{0, 1\} \times X$ and $Y \oplus Y = \{0, 1\} \times Y$. Then

$$((a, x), (b, y)) \in F(f) \Leftrightarrow a = b \wedge (x, y) \in f.$$

[1] Amazingly the categorical product is the disjoint union, too. SCS^* is self-dual.

Consider the initial object $\mathbf{1} = \{*\} \in (\mathsf{SCS}^*)^{\mathrm{ep}}$. The image under F is isomorphic to the discrete two-point set $F(\mathbf{1}) \cong \{0,1\}$. There is a unique ep pair $f : \mathbf{1} \to \{0,1\}$ which simply consists of the minimal relations

$$f^e = \mathbf{1} \times \{0,1\}, \quad f^p = \{0,1\} \times \mathbf{1}. \tag{5.3}$$

Observe that F preserves the separation axiom T_2. The fixed point of F when starting with the initial object will be a Hausdorff space, since it is realized as a subset of the product

$$\prod_{n \geq 0} F^n(\mathbf{1}).$$

5.1.2 A presentation for the Cantor set

We will use the following presentation for the Cantor set: Consider the space of binary sequences $\{0,1\}^{\mathbb{N}}$. We equip it with a metric

$$d(x,y) = \sum_{n \in \mathbb{N}} \frac{2}{3^n} |x_n - y_n|$$

where $|\cdot|$ is the usual absolute value. Then $C = \big(\{0,1\}^{\mathbb{N}}, d\big)$ is a metric space. Let $0 : \mathbb{N} \to \{0,1\}$ denote the zero map. C can be embedded into $\mathbb{R}$ by the map $i : x \mapsto d(x,0)$. The continuity of this map is a useful fact, since it provides that certain sets are closed. Let x be an element of C with $x_0, \ldots x_n$ the first $n+1$ terms and all other terms zero. Then

$$A = i^{-1}\left(\left[i(x), i(x) + \frac{1}{3^n}\right]\right)$$

is a closed set in C. Note that $\frac{1}{3^n} = \sum_{k=n+1}^{\infty} \frac{2}{3^k}$. Therefore if we complete $x_0, \ldots x_n$ with ones instead of zeros, we still get an element of A. Further, if we alter any element of the initial $n+1$ ones in x such that its value under i is bigger, then the distance to the old x is at least $\frac{2}{3^n}$. Hence

$$A = \left\{ y \in C \,\middle|\, \forall k \leq n : \, y_k = x_k \right\}. \tag{5.4}$$

5.1.3 The colimiting cocone

Consider the directed diagram $(F^n(\mathbf{1}),\, F^n(f))_{n \geq 0}$ in $(\mathsf{SCS}^*)^{\mathrm{ep}}$ where f is the unique morphism in (5.3). The n-th iterate $F^n(\mathbf{1})$ is the discrete set $\{0,1\}^n$. Define a map

$$p_n : C \to \{0,1\}^n, \quad p_n(x) = (x_0, \ldots, x_{n-1}).$$

For $n = 0$ we let $p_0 : x \mapsto *$ be the constant map. By what is stated in 5.1.2 the map p_n is continuous: Every point is closed in $\{0,1\}^n$, and every closed set is a finite union of points. Now the preimage of a point under p_n is precisely a set of the type (5.4), hence closed. Finite unions of such sets are closed in C, whence

p_n is continuos for all $n \in \mathbb{N}$. Further these maps are surjective. Since all the spaces in consideration are Hausdorff the p_n are perfect and therefore give rise to ep pairs $\mu_n : F^n(\mathbf{1}) \to C$. Finally note that $\mu_n^p \mu_n^e$ is the equivalence relation

$$(x, y) \in \mu_n^p \mu_n^e \Leftrightarrow \forall k < n : x_n = y_n.$$

The intersection (and therefore the supremum with respect to $\sqsubseteq$) of all these is the equal relation $=$. Hence μ is a colimiting cocone and C a fixed point for F, so

$$C \cong C \oplus C.$$

Outlook

The motivation for this theis was the idea to express certain topological spaces as a fixed point of an endofunctor. The Cantor set is only the most generic example, since the coproduct functor can be applied to any compact space. In general spaces which have a self-similar structure should be expressible as fixed points of functors reflecting this self-similarity. For instance, many universal covering spaces are pieced together using one or several basic components. Unfortunately the interesting examples are not compact and therefore cannot be accessed using the machinery in this thesis. Other compact examples like the Sierpinksi carpet lack a generating principle general enough to be a functor defined on the entire category of compact Hausdorff spaces. One suggestion to circumvent this problem is to enrich the spaces with additional order structure which tells the functor how to put the pieces together. Further some hypotheses made in this thesis may be stronger than neccessary, so some results may even apply to a bigger class of spaces.

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Biography

Born in 1982, the author was enrolled in a program of study called *Technomathematik* at the Universtiät Paderborn, Germany from 2002-2006. This thesis is the endpoint of a year in graduate school at Tulane University from 2006-2007.