Game Semantics for Dependent Types
Samson Abramsky, Radha Jagadeesan, Matthijs Vákár
February 2, 2016

Abstract
In [1], we presented a game semantics for dependent type theory (DTT), which we shall recall. It a model of DTT with Π-, 1-, Σ- and intensional Id-types, which is based on a slight variation of the (call-by-name) category of AJM-games and history-free winning well-bracketed strategies. The model satisfies Streicher’s criteria of intensionality and refutes function extensionality. The principle of uniqueness of identity proofs is satisfied. It contains a submodel as a full subcategory which gives a faithful model of DTT with Π-, 1-, Σ- and intensional Id-types and, additionally, finite inductive type families. This smaller model is fully (and faithfully) complete with respect to the syntax at the type hierarchy built without Id-types, as well as at the more general class of types where we allow for one strictly positive occurrence of an Id-type. Definability for the full type hierarchy with Id-types remains to be investigated.

The study of this model raised a few conceptual questions, relating to both type theory and game semantics (and their combination). Their study is currently work in progress, but we would like to share our preliminary thoughts as we believe they would serve as a good basis for discussion.

Firstly, the question rises how robust the model is under variation of the class of strategies considered. It turns out that the model is rather robust and, in fact, it is one of the few sources of non-trivial models which combine dependent types with effects (the other being the domain semantics of [2]). There seem to be no problems in dropping the bracketing condition. This corresponds to passing to DTT with control operators on ground types, from which we can define them on some but not all composite types [3], hence avoiding the inconsistency result of [4]. Neither do we see any problems in passing to innocent strategies in the style of [5, 6], which gives inductive families a more robust interpretation, eliminating to restrict to a subcategory to obtain a sound interpretation of their eliminators. Nor should there be any problems in passing to more general strategies where we drop the innocence condition but keep demanding visibility. This allows us to interpret ground store [7]. It remains to be investigated if we can generalise even further by dropping the visibility condition to interpret general references in the style of [8]. Dropping the winning condition on strategies or the determinism condition to interpret recursion and erratic choice in the style of [9] and [10] is mostly unproblematic as long as some care is taken with the flavour of identity types used.

This brings us to our second conceptual point: the notion of identity types used. In DTT, there are many possible choices in which axioms to demand for Id-types. The problem is that (intensional) Id-types, by definition are inductive families for which we demand a β- but not η-rule. By contrast with the situation of inductive definitions with an η-rule, this does not determine the structure uniquely. One option is to try and generate the simplest possible interpretation, which we did in [1]. Another is to additionally interpret useful practical principles like function extensionality. We will share some of our thoughts on how this can be done in game semantics.

Thirdly and finally, we believe there is a gap to be filled in the literature on a principled domain theoretic and game theoretic interpretation of strong type forming principles like (co)inductive families and induction-recursion. We believe this is due to their non-trivial nature in presence of effects. This has been highlighted recently by work describing frameworks for combining dependent types and computational effects [11, 12]. We believe this combination is still poorly understood with lots of fascinating questions to answer.

References


