

# Typed $\lambda$ -calculus: Further Questions

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The exam for this course consists of all exercises in Handouts 2–4, if you haven’t done them already, and two additional questions below.

## 1 Answers To “Concepts and Syntax” Exercises

Here are the answers to the exercises in Section 8 of Handout 1.

**Question** What integer is

```
let 3 be x.  
let inl  $\lambda y_{\mathbb{Z}}.(x + y)$  be u.  
let 4 be x.  
match u as {inl f.f2, inr f.0}
```

?

**Correct answer** 5

**Plausible but incorrect answer** 6

**Question** What integer is

```
let  $\lambda x_{\mathbb{Z}}. \text{inl } \lambda y_{\mathbb{Z}}.(x + y)$  be f.  
let f0 be u.  
match u as {  
  inl g. let f1 be v. match v as {inl h. g3, inr h. 0},  
  inr g. 0  
}
```

?

**Correct answer** 3

**Plausible but incorrect answer** 4. Both these exercises illustrate the idea of *static binding*, meaning that bindings cannot be changed. The incorrect answers, 6 and 4, are not in accordance with our definition of the notation. Unfortunately, Emacs Lisp would give you these answers. That is because it uses *dynamic binding*, meaning that a binding of  $x$  overwrites any previous binding of  $x$ .

**Question** (variant record type) For sets  $A, B, C, D, E$ , we define  $\alpha(A, B, C, D, E)$  to be the set of tuples

$$\{\langle \#left, x, y \rangle \mid x \in A, y \in B\} \cup \{\langle \#right, x, y, z \rangle \mid x \in C, y \in D, z \in E\}$$

Now think of  $\alpha$  as an operation on types. Inventing a reasonable syntax, given 2 introduction rules and 1 elimination rule for  $\alpha(A, B, C, D, E)$ .

**Answer** We invent the syntax

$$\langle \#left, M, N \rangle \qquad \langle \#right, M, N, P \rangle$$

for terms of type  $\alpha(A, B, C, D, E)$ . And we invent the syntax

$$\text{match } M \text{ as } \{\langle \#left, x, y \rangle. N, \langle \#right, x, y, z \rangle. N'\}$$

for pattern-matching a term  $M$  of type  $\alpha(A, B, C, D, E)$ .

The introduction rules are

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle \#left, M, N \rangle : \alpha(A, B, C, D, E)}$$

$$\frac{\Gamma \vdash M : C \quad \Gamma \vdash N : D \quad \Gamma \vdash P : E}{\Gamma \vdash \langle \#right, M, N, P \rangle : \alpha(A, B, C, D, E)}$$

The elimination rule is

$$\frac{\Gamma \vdash M : \alpha(A, B, C, D, E) \quad \Gamma, x : A, y : B \vdash N : F \quad \Gamma, x : C, y : D, z : E \vdash N' : F}{\Gamma \vdash \text{match } M \text{ as } \{\langle \#left, x, y \rangle. N, \langle \#right, x, y, z \rangle. N'\} : F}$$

**Question** For sets  $A, B, C, D, E, F, G$ , we define  $\beta(A, B, C, D, E, F, G)$  to be the set of functions that take

- a sequence of arguments  $(\#left, x, y)$ , where  $x \in A$  and  $y \in B$ , to an element of  $C$
- a sequence of arguments  $(\#right, x, y, z)$ , where  $x \in D$  and  $y \in E$  and  $z \in F$ , to an element of  $G$ .

Thus the first argument is always a tag, indicating how many other arguments there are, what their type is, and what the type of the result should be.

Now think of  $\beta$  as an operation on types. Inventing a reasonable syntax, give 1 introduction rule and 2 elimination rules for  $\beta(A, B, C, D, E, F, G)$ .

**Answer** We invent the syntax

$$\lambda\{(\#left, \mathbf{x}, \mathbf{y}) . M, (\#right, \mathbf{x}, \mathbf{y}, \mathbf{z}) . M'\}$$

for something of type  $\beta(A, B, C, D, E, F, G)$ . And we invent the syntax

$$M(\#left, N, N') \qquad M(\#right, N, N', N'')$$

for applying a term  $M$  of type  $\beta(A, B, C, D, E, F, G)$ .

The introduction rule is

$$\frac{\Gamma, \mathbf{x} : A, \mathbf{y} : B \vdash M : C \quad \Gamma, \mathbf{x} : D, \mathbf{y} : E, \mathbf{z} : F \vdash M' : G}{\Gamma \vdash \lambda\{(\#left, \mathbf{x}, \mathbf{y}) . M, (\#right, \mathbf{x}, \mathbf{y}, \mathbf{z}) . M'\} : \beta(A, B, C, D, E, F, G)}$$

The elimination rules are

$$\frac{\Gamma \vdash M : \beta(A, B, C, D, E, F, G) \quad \Gamma \vdash N : A \quad \Gamma \vdash N' : B}{\Gamma \vdash M(\#left, N, N') : C}$$

$$\frac{\Gamma \vdash M : \beta(A, B, C, D, E, F, G) \quad \Gamma \vdash N : D \quad \Gamma \vdash N' : E \quad \Gamma \vdash N'' : F}{\Gamma \vdash M(\#right, N, N', N'') : G}$$

## 2 Question on Pure $\lambda$ -calculus

This question is about the pure (i.e. no imperative features) simply typed  $\lambda$ -calculus.

In this language, a *syntactic isomorphism* from  $A$  to  $B$  consists of a term  $\mathbf{x} : A \vdash M : B$  and a term  $\mathbf{y} : B \vdash N : A$  such that the equations

$$\begin{aligned} \mathbf{x} : A \vdash N[M/\mathbf{x}] &= \mathbf{x} : B \\ \mathbf{y} : B \vdash M[N/\mathbf{x}] &= \mathbf{y} : A \end{aligned}$$

are provable in the equational theory. (NB This definition, as it stands, is not suitable for  $\lambda$ -calculus with imperative features.) Construct syntactic isomorphisms

$$\begin{aligned} (A + B) + C &\cong A + (B + C) \\ (A \times B) \rightarrow C &\cong A \rightarrow (B \rightarrow C) \\ (A + B) \rightarrow C &\cong (A \rightarrow C) \times (B \rightarrow C) \end{aligned}$$

### 3 Question on $\lambda$ -calculus with Imperative Features

CELL is a storage cell (piece of computer memory) that stores an integer.

Consider call-by-value  $\lambda$ -calculus, without divergence or printing, but with the facility to write to and read from CELL.

- `CELL := M. N`, where  $M$  is an integer expression. To evaluate this, first evaluate  $M$ , then put the answer in CELL (overwriting whatever was there previously), then evaluate  $N$ .
- `read CELL as x. N`. To evaluate this, define  $x$  to be whatever is currently in CELL, then evaluate  $N$ .

We write  $s, M \Downarrow s', T$  to mean that if  $M$  (a closed term) is evaluated at a time when CELL contains  $s$  (an integer), then it evaluates to  $T$  (a terminal term, with the same type as  $M$ ) with CELL then containing  $s'$  (an integer). Give an inductive definition of the relation  $\Downarrow$ .