Modal properties of recursive programs

Work in progress

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Summary

- A language
- Roscoe’s *Seeing Beyond Divergence* model
- A model of lower bisimilarity
- What’s really happening: modal logic
A basic language

Syntax

Let $\mathcal{A}$ be a countable alphabet.

\[
M ::= \text{print } c.\ M \mid x \mid \text{rec } x.\ M \mid \text{choose}_{n \in \mathbb{N}}\ M_n \quad c \in \mathcal{A}
\]

Many other things can be added.
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Many other things can be added.

Small-step semantics

\[
\begin{align*}
\text{print } c.\ M & \rightsquigarrow M \\
\text{rec } x.\ M & \rightsquigarrow M[\text{rec } x.\ M/x] \\
\text{choose}_{n \in \mathbb{N}}\ M_n & \rightsquigarrow M^{\hat{n}} \quad \hat{n} \in \mathbb{N}
\end{align*}
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A basic language

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Let $\mathcal{A}$ be a countable alphabet.

$$M ::= \text{print } c. \, M \mid x \mid \text{rec } x. \, M \mid \text{choose}_{n \in \mathbb{N}} \, M_n \quad c \in \mathcal{A}$$

Many other things can be added.

Small-step semantics

$$\begin{align*}
\text{print } c. \, M & \leadsto M \\
\text{rec } x. \, M & \leadsto M[\text{rec } x. \, M/x] \\
\text{choose}_{n \in \mathbb{N}} \, M_n & \leadsto M_{\hat{n}} \quad \hat{n} \in \mathbb{N}
\end{align*}$$

A program either
- prints a finite string, then diverges
- or prints an infinite string.
Medium step semantics

**Convergence** \( M \xrightarrow{c} N \) defined inductively

\[
\begin{align*}
\text{print } c. & \quad M \xrightarrow{c} M \\
M^\hat{n} & \xrightarrow{c} N \\
\text{choose } \, \forall n \in \mathbb{N} & \quad M_n \xrightarrow{c} N
\end{align*}
\]

\[
M[\text{rec } x. \, M/x] \xrightarrow{c} N
\]

**Divergence** \( M \xuparrow \) defined coinductively

\[
\begin{align*}
\text{rec } x. & \quad M \xuparrow \\
M^\hat{n} & \xuparrow \\
\text{choose } \, \forall n \in \mathbb{N} & \quad M_n \xuparrow
\end{align*}
\]

\[
M[\text{rec } x. \, M/x] \xuparrow
\]

\[
\hat{n} \in \mathbb{N}
\]
Medium step semantics

Convergence $M \xrightarrow{c} N$ defined inductively

- $\text{print } c.\ M \xrightarrow{c} M$
- $M_{\hat{n}} \xrightarrow{c} N$
- $\text{choose}_{n \in \mathbb{N}} M_n \xrightarrow{c} N \quad \hat{n} \in \mathbb{N}$

Divergence $M \uparrow$ defined coinductively

- $M[\text{rec } x.\ M/x] \uparrow$
- $M_{\hat{n}} \uparrow$
- $\text{choose}_{n \in \mathbb{N}} M_n \uparrow \quad \hat{n} \in \mathbb{N}$

We have

- $M \xleftarrow{c} N$ iff $M \xrightarrow{\sim \star} \xrightarrow{c} N$
- $M \uparrow$ iff $M \xrightarrow{\sim \omega}$
Well-pointed semantics

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To interpret recursion, we need an appropriate way of finding a fixpoint of an endofunction.

For example, least fixpoint or greatest fixpoint.

Roscoe’s Seeing Beyond Divergence model uses a reflected fixpoint.
A closed command $M$ has

- a set $T(M) \subseteq A^*$ of finite traces

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Finite traces, divergences, and infinite traces

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A closed command $M$ has

- a set $T(M) \subseteq \mathcal{A}^*$ of finite traces
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  $M$ can print \texttt{hello} then diverge

- a set $I(M) \subseteq \mathcal{A}^\omega$ of infinite traces
  $M$ can print \texttt{helloworldworldworld} \ldots
Seeing Beyond Divergence (1)

Definition of $[M]_{N}$

- the set of finite traces of $M$, together with extensions of divergences
- the set of extensions of divergences of $M$
- the set of infinite traces of $M$, together with extensions of divergences

This semantics is divergence strict.
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This semantics is divergence strict.

To model recursion, we take the greatest fixpoint. (Reverse ordering is the upper powerdomain.)
Definition of $[M]_{\text{SBD}}$

- the set of finite traces of $M$
- the set of divergences of $M$
- the set of infinite traces, together with limits of divergences (called \(\omega\)-divergences)
Definition of $[M]_{\text{SBD}}$

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To model recursion:

- first compute the greatest fixpoint wrt $[]_\mathcal{N}$, giving a “diamond”: a complete lattice of possible solutions that are $[]_\mathcal{N}$ equivalent
- then compute the least fixpoint wrt $[]_{\text{SBD}}$ within that complete lattice.

This is called the reflected fixpoint.
### Continuity

For any recursion, the second phase of fixpoint calculation converges in $\omega$ steps.

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Lexicographic Ordering

The reflected fixpoint is the least prefixed point wrt the lexicographic ordering:

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Lexicographic Ordering
The reflected fixpoint is the least prefixed point wrt the lexicographic ordering:
- reverse inclusion for $[\_]_N$
- then inclusion for $[\_]_{SBD}$
Terms are not monotone wrt this ordering.
Let $\mathcal{R}$ be a binary relation on closed terms. It is a **lower simulation** when $M \mathcal{R} M'$ and $M \trianglerighteq\downarrow N$ implies $\exists N'$ such that $M' \trianglerighteq\downarrow N'$ and $N \mathcal{R} N'$.

It is a **lower bisimulation** when $\mathcal{R}$ and $\mathcal{R}^{\text{op}}$ are lower simulations.

It is a **convex bisimulation** when moreover $M \mathcal{R} M'$ implies $M \uparrow\iff M' \uparrow$.

The greatest lower bisimulation is called **lower bisimilarity**.
Lower And Convex Bisimulation

Let $\mathcal{R}$ be a binary relation on closed terms.

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  (Opponent moves first, and in each move can play either left or right)
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Two closed terms $M, M'$ are lower bisimilar
- iff they have the same anamorphic image
- iff there is a strategy for the **bisimilarity game** between them (Opponent moves first, and in each move can play either left or right)
- iff they satisfy the same formulas in **Hennessy-Milner logic**

$$P ::= \Box a.P \mid \bigvee_{j \in J} P_j \mid \neg P$$
Model of bisimilarity: nested simulation

A 2-nested (lower) simulation is a simulation contained in mutual similarity.

The intersection of $n$-nested similarity for $n < \omega_1$ is lower bisimilarity.

Theorem: Up to lower bisimilarity, $\text{rec } x.M$ is the lexicographically least prefixed point for $N \mapsto N[\text{rec } x.M/x]$ wrt this sequence of precongruences.
A 2-nested (lower) simulation is a simulation contained in mutual similarity.

Two closed terms $M, M'$ are related by 2-nested similarity

- iff there is a strategy for the 2-nested simulation game (Opponent starts on the left, and can switch once)
- iff $M \models P$ implies $M' \models P$ whenever $P$ has at most one level of negation.

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A **3-nested** lower simulation is a simulation contained in mutual 2-nested similarity. And so through all countable ordinals.
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M \models P \implies M' \models P
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A 3-nested lower simulation is a simulation contained in mutual 2-nested similarity. And so through all countable ordinals.

The intersection of \( n \)-nested similarity for \( n < \omega_1 \) is lower bisimilarity.

**Theorem** Up to lower bisimilarity, \( \text{rec } x. \ M \) is the lexicographically least prefixed point for \( N \leftrightarrow N[\text{rec } x. \ M/x] \) wrt this sequence of precongruences.
Digression: other models of bisimilarity (1)

Synchronization Trees

Milner and Winskel have studied semantics in which
- a closed term denotes a synchronization tree of possible behaviours.
- an open term denotes a function (actually a functor) from synchronization trees to synchronization trees.

Very intensional: the idempotency law $M \mathbin{\text{or}} M = M$ is not validated.
Synchronization Trees

Milner and Winskel have studied semantics in which

- a closed term denotes a \textit{synchronization tree} of possible behaviours.
- an open term denotes a function \textit{(actually a functor)} from synchronization trees to synchronization trees.

Very intensional: the idempotency law $M \text{ or } M = M$ is not validated.

Lower bisimilarity is studied as a relation on the trees, but this is not part of the denotational semantics.
Abramsky’s domain equation for bisimulation

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Abramsky’s domain equation for bisimulation

Abramsky gave a semantics (for finite nondeterminism) using a domain equation involving the convex powerdomain.

For nondivergent terms, denotational equivalence coincides with lower bisimilarity.

But in general, terms may have the same denotation without being lower bisimilar.

This is inevitable in least fixpoint semantics.
Some general points

We want to understand various denotational models of nondeterministic languages with recursion.
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A recursion is interpreted, in these models, as the least prefixed point wrt the induced lexicographic partial order.

**Warning** Terms are not even monotone wrt this lexicographic partial order.
Modal logic with may and must

Modal logic in the style of Hennessy-Milner:

\[ A ::= \neg A \mid \bigvee_{i \in I} A_i \mid \bigwedge_{i \in I} A_i \mid \Diamond a.A \mid \Box_{s \in A^*} A_s \]

where \( I \) is bounded by some suitable cardinal.
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**Meaning of \( \Diamond \)**

\( \Diamond a.A \) means it is possible that \( a \) will be printed and then \( A \) will be satisfied.
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\( \diamond a.A \) means it is possible that \( a \) will be printed and then \( A \) will be satisfied.

**Meaning of \( \Box \)**

\( \Box_{s \in A^*} A_s \) means a time will come when \( A_s \) will be satisfied, where \( s \) is the string printed between now and then.
We say $M \preceq_{(A)} M'$ when

- for every context $C$, if $C[M] \models A$ then $C[M'] \vdash A$. 
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This is a preorder, and we can speak of the $\preceq (A)$ equivalence class of $M$. 

\[ \text{Paul Blain Levy (University of Birmingham)} \] 

\[ \text{Modal properties of recursive programs} \] 

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Contextual preorders

We say $M \preceq (A) M'$ when

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This is a preorder, and we can speak of the $\preceq (A)$ equivalence class of $M$.

More generally, we say $M \preceq (\{A_i\}_{i \in I}) M$ when

- for every context $C$ and $i \in I$, if $C[M] \models A_i$ then $C[M'] \models A_i$. 
When does a recursive program satisfy $\Diamond$?

We want to know when $C[\text{rec } x. M]$ satisfies $B \overset{\text{def}}{=} \Diamond a.A$. 

$U$ be the equivalence class of $\text{rec } x. M$. Clearly $\theta_M : N \mapsto M[N/x]$ is an endofunction on $U$, monotone wrt $\preceq$. 

Theorem Suppose $U$ has a $\preceq(B)$ least element of $U$, call it $N$. Then $C[\text{rec } x. M] \models B \iff$ there exists $n \in N$ such that $C[\theta_n M N] \models a.A$. 

The special case that $A = \text{True}$ gives lower powerdomain semantics.
When does a recursive program satisfy $\Diamond$?

We want to know when $\mathcal{C}[\text{rec } x. \ M]$ satisfies $B \overset{\text{def}}{=} \Diamond a.A.$

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Theorem

Suppose $U$ has a $\preceq (B)$ least element of $U$, call it $N$.

Then $C[\text{rec } x. \ M] \models \diamond a.A$ iff there exists $n \in \mathbb{N}$ such that $C[\theta^n_M N] \models \diamond a.A$. 

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When does a recursive program satisfy $□$?

We want to know when $C[\text{rec } x. \ M]$ satisfies $B \overset{\text{def}}{=} \bigcirc_{s \in A^*A_s}$. 

Let $U_s$ be the $\equiv (A_s)$ equivalence class of $\text{rec } x. \ M$. 

$\theta_M : N \mapsto M[N/\text{x}]$ is an endofunction on $\bigcap \ s \in A^*U_s$ monotone wrt $\equiv (B)$.

Conjecture

Define a sequence $(N_\alpha)_{\alpha < \omega}$ contained in $U_s$, increasing wrt $\equiv (B)$. 

$N_0 \overset{\text{def}}{=} \equiv (B)$ least element of $\bigcap \ s \in A^*U_s$, assuming it exists. 

$N_{\alpha+1} \overset{\text{def}}{=} \theta_M N_\beta$ 

$N_\gamma \overset{\text{def}}{=} \equiv (B)$ supremum of $(N_\alpha)_{\alpha < \gamma}$, assuming it exists.

Then $C[\text{rec } x. \ M] \models B$ iff there exists $\alpha < \omega$ such that $C[N_\alpha] \models B$. 

The special case that $A = \text{True}$ gives upper powerdomain semantics.
When does a recursive program satisfy $\square$?

We want to know when $C[\text{rec } x. M]$ satisfies $B \overset{\text{def}}{=} \Box_{s \in A^*} A_s$.

Let $U_s$ be the $\preceq (A_s)$ equivalence class of $\text{rec } x.M$.

$\theta_M : N \mapsto M[N/x]$ is an endofunction on $\bigcap_{s \in A^*} U_s$ monotone wrt $\preceq (B)$. 
When does a recursive program satisfy $\square$?

We want to know when $C[\text{rec } x. \ M]$ satisfies $B \overset{\text{def}}{=} \square_{s \in A^*} A_s$.

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**Conjecture**

Define a sequence $(N_\alpha)_{\alpha < \omega_1}$ contained in $U$, increasing wrt $\preceq (B)$

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\begin{align*}
N_0 & \overset{\text{def}}{=} \preceq (B) \text{ least element of } \bigcap_{s \in A^*} U_s, \text{ assuming it exists} \\
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- $N_\gamma \overset{\text{def}}{=} \preceq (B)$ supremum of $(N_\alpha)_{\alpha < \gamma}$, assuming it exists

Then $C[\text{rec } x. M] \models B$ iff there exists $\alpha < \omega_1$ such that $C[N_\alpha] \models B$. 
When does a recursive program satisfy $\square$?

We want to know when $C[\text{rec } x. \ M]$ satisfies $B \overset{\text{def}}{=} \Box_{s \in A^*} A_s$.

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Then $C[\text{rec } x. \ M] \models B$ iff there exists $\alpha < \omega_1$ such that $C[N_\alpha] \models B$.

The special case that $A = \text{True}$ gives upper powerdomain semantics.
Let \( t \) be a finite trace, divergence or infinite trace, and let \( s \) be a finite prefix.

We say that \( M \) semi-validates \( s \sqsubseteq t \) when it is possible for \( M \) to prove \( s \) and never refute \( t \).

Two terms \( M, M' \) are SBD equivalent when they semi-validate the same prefixes.

\( M \) validates \( \text{hello} \sqsubseteq \text{helloworldworldworld...} \) when \( M \models 3 \text{hello} \).\( \neg 2 s \). \( s \not\sqsubseteq \text{worldworldworld...} \).
Let $t$ be a finite trace, divergence or infinite trace, and let $s$ be a finite prefix.

We say that $M$ semi-validates $s \sqsubseteq t$ when it is possible for $M$ to prove $s$ and never refute $t$.

Two terms $M, M'$ are SBD equivalent when they semi-validate the same prefixes.
Let $t$ be a finite trace, divergence or infinite trace, and let $s$ be a finite prefix.

We say that $M$ semi-validates $s \sqsubseteq t$ when it is possible for $M$ to prove $s$ and never refute $t$.

Two terms $M, M'$ are SBD equivalent when they semi-validate the same prefixes.

$M$ validates $\text{hello} \sqsubseteq \text{helloworldworldworld...}$ when

$$M \vDash \Diamond \text{hello}. \neg \Box s.s \not\sqsubseteq \text{worldworldworld...}$$

We can now see how the two-part calculation of a recursion arises.
Directions

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Studying this syntactically is awkward, since

- we do not know what the \( \preceq (A) \) preorders actually are
- the required suprema might not exist.
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Perhaps a deductively closed set of modal formulas could serve as a generalized program?
To define big-step semantics of a functional language (even one with McCarthy’s amb):

- first define convergence (⇓) as a least prefixed point
- then define divergence (⇑) as a greatest postfixed point.
Another lexicographically least prefixed point

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Big-step semantics can be seen as describing the denotational semantics of an interpreter, which is a first-order recursive program.
To define big-step semantics of a functional language (even one with McCarthy’s amb):

- first define convergence \((\Downarrow)\) as a least prefixed point
- then define divergence \((\Uparrow)\) as a greatest postfixed point.

Big-step semantics can be seen as describing the denotational semantics of an interpreter, which is a first-order recursive program.

The pair \((\Downarrow, \Uparrow)\) is a lexicographically least prefixed point.