Morphisms between plays

Paul Blain Levy

University of Birmingham

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In a small category $A$

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We form a category whose objects are plays on $R$. 

Examples:

1. Innocence
2. Single-threadedness
3. Asynchronous games

The first two examples use alternating plays.
The point of the talk

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Examples:

1. Innocence
2. Single-threadedness
3. Asynchronous games

The first two examples use alternating plays.
A play $s$ consists of

- an initial segment moves $s \subseteq \mathbb{N}$
- (justification pointers) a map moves $s \rightarrow \{\ast\} +$ moves $s$
- (arena elements) a map moves $s \rightarrow R$ (arena element).

with the usual conditions.
A strategy $\sigma$ (Opponent first) on $R$ is a set of even-length plays that
- is lower wrt the prefix order
- contains the empty play
- is deterministic.
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**Traces and divergences**

- **Finite trace** is $s \in \sigma$
- **Divergence** is $sm \perp$ where $s \in \sigma$ but $\not\exists n. \ smn \in \sigma$.
- **Infinite trace** is infinite play whose even-length prefixes are all in $\sigma$. 
Innocent Strategies

- A **P-visible play** is one in which Proponent always points to a move in the current P-view.
- A **P-visible strategy** is one in which all plays are P-visible.
- An **innocent** strategy is a special kind of P-visible strategy.
- It can be represented as a set of P-views. (After the first move, Opponent always points to the previous move.)

Innocent strategies are used to model **state-free terms**.
A thread-visible play is one in which Proponent doesn’t change thread.

A thread-visible strategy is one in which all plays are thread-visible.

A single-threaded strategy is a special kind of thread-visible strategy.

It can be represented as a set of well-opened plays. (After the first move, Opponent never again points to ∗).

Single-threaded strategies are used to model values.
Some problems

Traditional innocence

Doesn’t work for certain language features:

- Nondeterminism, under linear time semantics
- Name generation ($\nu$-calculus)
- Polymorphism

In each case there are state-free terms whose denotation isn’t “innocent”.

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Traditional innocence

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Traditional single-threadedness

Doesn’t work for polymorphism.
There are (state-free) values whose denotation isn’t “single-threaded”.

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A category for innocence

We form a category of P-visible plays and P-viewing morphisms.

P-viewing morphism \( s \xrightarrow{f} t \)

- A function moves \( s \rightarrow \) moves \( t \).
- Preserves arena elements and justification pointers.
- If \( n^O \) is followed by \( m^P \) then \( f(n) \) is followed by \( f(m) \).
- If \( n^O \) is followed by \( \bot \) then \( f(n) \) is followed by \( \bot \).
Theorem

A P-visible strategy is innocent iff its finite trace set and its divergence set are lower with respect to P-viewing morphisms.

(Cf. Melliès' paper: "Asynchronous Games 2: the true concurrency of innocence").

There's an analogous theorem for OP-visible strategies.
Theorem

A P-visible strategy is innocent iff both its finite trace set and its divergence set are lower wrt P-viewing morphisms.

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A P-visible strategy is innocent iff both its finite trace set and its divergence set are lower wrt P-viewing morphisms.

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Characterizing innocence

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Adapt the category to other settings.

In particular names (*f should preserve and reflect name equality*) and polymorphism.
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In these settings, define an innocent strategy to be a P-visible strategy whose finite trace set and divergence set are both lower wrt P-viewing morphisms.
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Likewise, working up to may-testing, we define an innocent nondeterministic strategy to be a set of P-visible plays lower wrt P-viewing morphisms and containing the empty play.
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Likewise, working up to may-testing, we define an innocent nondeterministic strategy to be a set of P-visible plays lower wrt P-viewing morphisms and containing the empty play.

Conjecture: these are precisely the strategies definable by a state-free term.
We form a category of thread-visible plays and threading morphisms.

**Threading morphism** \( s \xrightarrow{f} t \)

- A function moves \( s \rightarrow \rightarrow \text{moves } t \).
- Preserves arena elements and justification pointers.
- If \( n^O \) is followed by \( m^P \) then \( f(n) \) is followed by \( f(m) \).
- If \( n^O \) is followed by \( \bot \) then \( f(n) \) is followed by \( \bot \).
- If \( m^P \) is thread-followed by \( n^O \) (there might be moves in between, but not from the same thread) then \( f(m) \) is thread-followed by \( f(n) \).
Theorem

A thread-visible strategy is single-threaded iff
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A thread-visible strategy is single-threaded iff both its finite trace set and its divergence set are lower wrt threading morphisms.
Adapt the category to other settings.

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In this setting, define a single-threaded strategy in that setting: a thread-visible strategy whose finite trace set and divergence set are both lower wrt threading morphisms.
Adapt the category to other settings.

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Conjecture: these are precisely the strategies definable by a value.
Asynchronous Games

Laird, Ghica, Murawski gave a game semantics for concurrent calculi, up to may-testing, as follows.

A strategy $\sigma$ on $R$ is a set of non-alternating plays

- lower wrt the prefix order
- containing the empty play
- (P-swap) if $sm^n t \in \sigma$ then $snm^P t \in \sigma$ (assuming it’s a play)
- (O-swap) if $smn^O t \in \sigma$ then $sm^O nt \in \sigma$ (assuming it’s a play)
- (O-completeness) if $s \in \sigma$ then $sm^O \in \sigma$. 

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Problems with swap formulation

- Challenging to prove basic categorical properties, e.g. left and right identity laws.
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- Challenging to prove basic categorical properties, e.g. left and right identity laws.
- Challenging to adapt to infinite plays, where we might want to perform infinitely many swaps simultaneously.
We form a category of non-alternating plays and asynchrony morphisms.

**Asynchrony morphism**  
\[ s \xrightarrow{f} t \]

- A partial injection moves \( s \rightarrow \) moves \( t \).
- Preserves arena elements and justification pointers.
- For any \( m^P \in \text{moves } s \),
  - \( f(m) \) is defined
  - for any \( n^O < f(m) \), there is \( n' < m \) such that \( f(n') = n \).
Theorem

A set of non-alternating plays is an asynchronous strategy iff it is lower wrt asynchrony morphisms and contains the empty play.
Characterizing asynchronous strategies

Theorem

A set of non-alternating plays is an asynchronous strategy iff it is lower wrt asynchrony morphisms and contains the empty play.

A similar result was proved by Mohamed Menaa, and used to establish the categorical properties of the asynchronous game model.