Mandelbrot and Julia Sets

Peter Tiño

P.Tino@cs.bham.ac.uk, www.cs.bham.ac.uk/~pxt

School of Computer Science
University of Birmingham, UK
Complex numbers

‘Imaginary’ number: $i = \sqrt{-1}$
Note: $i^2 = -1$

Complex numbers: $z = a + i \cdot b$
Each complex number $z = a + i \cdot b$
can be represented as a 2-dimensional point $(a, b) \in \mathbb{R}^2$.

$z_1 = a_1 + i \cdot b_1$, $z_2 = a_2 + i \cdot b_2$

Addition: $z_1 + z_2 = (a_1 + a_2) + i \cdot (b_1 + b_2)$

Multiplication: $z_1 \cdot z_2 = (a_1 \cdot a_2 - b_1 \cdot b_2) + i \cdot (a_1 \cdot b_2 + a_2 \cdot b_1)$

Modulus (‘size’): $|z| = z \cdot z = \sqrt{a^2 + b^2}$
Complex numbers

\[ z = a + i \cdot b \]

\[ z_j, \ j = 1, 2, 3, 4, \] are solutions of \( z^4 = 1. \)
Mandelbrot set

Pick a complex number $c$.

Start at the origin: $z_0 = 0 + i \cdot 0$
and iterate:

$$z_{n+1} = z_n^2 + c.$$ 

Key question:
Will the trajectory $\{z_n\}_{n=0}^{\infty}$ diverge, or will it stay forever inside a bounded set?

Different answer for different $c$!

The Mandelbrot set contains those complex numbers $c$ for which the trajectory $\{z_n\}_{n=0}^{\infty}$ remains bounded.

Actually, it can be shown that if at some stage $|z_n| > 2$, then the trajectory will diverge.
Mandelbrot set - algorithm

For each complex number $c$ in a subset of complex plane

1. Set $z_0 = 0 + i \cdot 0$
2. For $n = 0 : N_{max}$
   - Compute $z_{n+1} = z_n^2 + c$
   - If $|z_{n+1}| > 2$, then break out of loop
3. If $n < N_{max}$, then color point $c$ white
   (trajectory diverges)
4. If $n = N_{max}$, then color point $c$ black
   (trajectory stays bounded)
Mandelbrot set - picture
Explorations of Mandelbrot Set

When zooming into the Mandelbrot set we can discover peculiar shapes and amazing forms of self-similarity!

It is often instructive to color code each $c$ not in the Mandelbrot set (e.g. those that give rise to a divergent trajectory) with the "escape rate" - how many iterations it takes to escape from the centered disk of radius 2, starting at the origin.

Explore your own color schemes for the ‘escape rates’ and zoom into different regions of the Mandelbrot set.
Mandelbrot set - linear color scheme
Mandelbrot set - square color scheme
Mandelbrot set - log color scheme
Mandelbrot set - linear ‘mirror’ color scheme
Mandelbrot set - square ‘mirror’ color scheme
Mandelbrot set - log ‘mirror’ color scheme
Example of zooming in - log ‘mirror’ color scheme
Zooming in - level 1
Zooming in - level 2
Zooming in - level 3

Mandelbrot set – level 3

Mandelbrot and Julia Sets – p.17/25
Zooming in - another example
Nice!
There is only one Mandelbrot set. We always start iterations in $z_0 = 0 + i \cdot 0$, and ask for each $c$, whether $z_{n+1} = z_n^2 + c$ diverges, or not.

Now, we will have many different Julia sets, one for each $c$.

For a given complex number $c$, investigate for which starting positions $z_0$ will $z_{n+1} = z_n^2 + c$ diverge and for which $z_0$ it will stay confined to a bounded region of the complex plane.

Julia set associated with a complex number $c$ contains those initial positions $z_0$ for which the trajectory of $z_{n+1} = z_n^2 + c$ remains bounded.
For each complex number $z_0$ in a subset of complex plane

1. For $n = 0 : N_{max}$
   - Compute $z_{n+1} = z_n^2 + c$
   - If $|z_{n+1}| > 2$, then break out of loop

2. If $n < N_{max}$, then color point $z_0$ white (trajectory diverges)

3. If $n = N_{max}$, then color point $z_0$ black (trajectory stays bounded)

As with Mandelbrot set, one can use different color schemes for initial conditions not in the Julia set.
Julia sets - pictures
A Julia set - log ‘mirror’ color scheme

\[ c = 0.277334 - i \cdot 0.00742 \]
Zooming in - level 1
Zooming in - level 2

Julia set – level 2