Aspects of GEOMETRIC LOGIC

Talk given at Workshop
Logic, Categories, Semantics
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- Logic, Categorical semantics
- USFs:
  - Geometric types
  - Ontology - observational
  - Geometricity = continuity
  - Fibrewise topology (bundles)

Geometric logic

- theories - must use sequent
- examples - algebraic
- inference rules

Geometric logic

First order, many sorted, positive, infinitary
Signature Σ: Sorts, functions, predicates
Formulae φ: use \( \top, \land, \bot, \lor, =, \exists \)

Formal functions in context (\( \Gamma, \phi \))
finite list of sorted variables
All free variables are in \( \Gamma \)

Sequent \( \phi \vdash \psi \) (\( \Sigma, \phi \), (\( \psi, \chi \)) both formulae in context

\( \forall \Gamma (\phi \rightarrow \psi) \)

Example I: Reals

Signature: no sorts
propositions (nullary predicates) \( P_\varphi \)

Topology: \( P \in \) open interval \( (q, r) \)

Axioms:
\( P_{q} \land P_{q'} \vdash \forall v \{ P_{st} \mid \max(v, q) < s < t < \min(v, r) \} \)
\( \top \vdash \forall v \{ P_{q - 3, q + 3} \mid q \in \mathbb{Q} \} \)

read \( v \in \mathbb{Q} \land \top \land \top \Rightarrow \top \)
Example II: Commutative rings

Signature: sort $\rightarrow R$
functions: $0, 1 : 1 \rightarrow R$
$-$ : $R \rightarrow R$
$+ : R^{2} \rightarrow R$

Axioms:
$T \xrightarrow{\otimes} x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$T \xrightarrow{\otimes} x + (y + z) = (x + y) + z$
$T \xrightarrow{\otimes} x \cdot 1 = x$
$T \xrightarrow{\otimes} 1 \cdot x = x$

etc.

$T \xrightarrow{\otimes} x \cdot y = y \cdot x$
$T \xrightarrow{\otimes} x + 0 = x$
$T \xrightarrow{\otimes} x + (-x) = 0$

Invertibles form complement of proper ideal

Example III: Commutative local rings

Signature: same as commutative rings

Axioms: same as commutative rings

$+ (\exists y) x + y \cdot y = 1 \wedge (\exists y) x \cdot y = y \cdot x \wedge 0 = 1 \wedge 1$

Invertibles form complement of a proper ideal

Follow account in Elephant

Inference rules I: Propositional

Sequent based: because no $\Rightarrow$
No other surprises

\[ \frac{\phi \vdash \psi}{\exists \xi (\phi \land \psi) \vdash \exists \xi \phi \land \exists \xi \psi} \]

\[ \frac{\phi \vdash \psi}{\psi \vdash \phi} \]

\[ \frac{\phi \vdash \psi, \xi \vdash \eta}{\phi, \xi \vdash \psi \land \eta} \]

\[ \frac{\phi \vdash \psi, \xi \vdash \eta}{\phi, \xi \vdash \psi, \eta \vdash \eta} \]

Inference rules II: Predicate

Always assuming formulae in correct contexts

\[ \phi \vdash \psi \]

\[ \psi \vdash \phi \]

\[ \exists \xi (\phi \land \psi) \vdash \exists \xi \phi \land \exists \xi \psi \]

\[ \psi \vdash \phi \]

\[ \phi \vdash \exists \xi \phi \]

Need this in absence of $\exists$
Again no surprises except \( (\forall x) \phi \)

Suppose have \( T \vdash x \phi \) as axiom

Deduce:

\[
\begin{array}{c}
\forall x \phi \\
\phi \vdash (\exists x) \phi \\
\hline
T \vdash (\exists x) \phi
\end{array}
\]

even though no free variable

CANNOT conclude \( T \vdash (\exists x) \phi \)

Rules work correctly for empty carriers

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**Categorical semantics**

- models in Grothendieck toposes
- axiom of unique choice
- geometric types
- geometricity of constructions

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**Semantics - categorical**

Syntax
- sort
- sort tuple

In context {term, formula, \( \wedge \), \( \vee \), \( \exists \), \( \forall \), sequent}

Interpretation
- object = carrier
- product
- morphism
- subobject
- pullback
- equalizer
- image
- image of coproduct
- truth value
- (order relation between subobjects)

Again, see Elephant

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**Categorical structure required**

- geometric category
- to interpret classical logic & its rules validly

In practice want more: Grothendieck topos

In particular, often want unique choice
- non-logical principle
- every total, single-valued relations defines a morphism, "balanced"
- categorically: monic epis are isos

Can use less: arithmetic universe "closed"
- if disjunctions countable and internalizable as \( \exists \)
- \( \forall n \in N \phi(n) \rightarrow (\exists n : N) \phi(n, n) \)
Geometric types
- characterized uniquely up to iso
  by geometric structure axioms
- geometric v. non-geometric constructions
  in Grothendieck toposes

Geometric types
characterized uniquely up to iso
by geometric structure axioms

E.g., sorts \( A, L \),
Functions \( \text{nil}: 1 \to L \), \( \text{cons}: A \times L \to L \)
Axioms
\[
\text{cons} (a, \text{nil}) = \text{nil} + \_a, L
\]
\[
\text{cons} (a, \text{cons}(x, l)) = \text{cons}(a, \text{cons}(x, l)) + \_a, L
\]
\[
T = \forall x \exists a \in \mathbb{A} \, \exists b \in \mathbb{B} \, \exists c \in \mathbb{C} \, \cdots \exists n \in \mathbb{N} \, \cdots \text{abbreviate } [a_0, \ldots, a_m]
\]

Recursively defined family of formulae, indexed by

Interpretation of \( L \) has to be parametrized list
object of that of \( A \)

Impossible with finitary logic

Proof sketch
Given functions \( f : B \to Y \)
want unique \( r = \text{rec}(f, g) : L \times B \to Y \)

Logically define graph of \( r \), \( Y = L \times B \times Y \)

Prove \( Y \) total & single valued
Appeal to unique choice to get \( r \)
Prove uniqueness of \( r \)

Geometric constructions
- e.g.
  finite limits,
  set-indexed colimit
  \{ cf. Giraud’s theorem, characterizing Grothendieck toposes
  free algebras - e.g. \( \mathbb{N} \), list object
  finite power set = free semilattice
also get finitely bounded \( A \)

Non-geometric constructions
  exponentials (function types)
\( \mathbb{N} \), power set, \( \mathbb{P}(X) \)

\( X, R, C \) (as sets)

Various kinds
Geometric types: two views

Syntactic sugar
Nice but not strictly necessary
Can do it all with infinite disjunctions

Syntactic sugar

Improve foundations
Avoid dependence on external infinities (at least for countable V)

Useful either way

Arithmetic universes

Example: Whereas

Sorts: none

Predicates: L, R ∈ \text{a}

Axioms:
\[ L(q), L(q') \|
\[ R(q), R(q') \|
\[ q < r \implies (q < q' \lor q' < r) \]
\[ q < r \implies (q < r' \lor r < r') \]
\[ L(q) \land R(q) \implies q < r \]

- Directly describes Dedekind sections
- Equivalent to propositional version

Ontology:

- Matching logic to what you're talking about

Geometric logic ↔ observational ontology

Formula — finitely observable property
\[ \land \quad \text{observe all conjuncts} \]
\[ \lor \quad \text{observe one disjunct} \]
\[ \neg, \to \quad \text{no observational account} \]

Sequent — background assumption
- scientific hypothesis

Ontology
- observational
- sets: existence + equality of elements
**Popperian Rejection**

Suppose -
- Theory T includes some axioms \( \Phi \vdash \bot \)
- Experimental observations \( E \) expressed as axioms \( T \vdash \Phi \)
- in \( T \cup E \) can infer \( T \vdash \bot \)

Then theory T is refuted by experimental results, \( E \).

**Predicate Ontology**

Observations "serendipitous"

Serendipity = the faculty of making happy chance finds

- How can you know that you have apprehended an element?
- How can you know that two elements are equal?

E.g. \( G \) a finitely presented group
- To apprehend element: write word in generator
- To affirm equality: find proof from relations

**Ontology of \( F \)**

\((\exists y) \phi(x,y) \land x : x, y : y\)

To apprehend element:
- apprehend \( x, y \) & affirm \( \phi(x,y) \)

To find equality \( \langle x, y \rangle = \langle x, y' \rangle \), affirm \( x = x' \)

**Ontology of \( F \)**

\( \forall (x) \rightarrow (\exists y) \phi(x,y) \)

If you have apprehended \( x \) then
- You already have \( y \)
- There is a \( y \) somewhere

**Ontology of List Objects**

Given observable set \( A \):
- To apprehend element of List(\( A \)) -
  - get a natural number \( n \) and, for each \( 0 \leq i \leq n \)
  - apprehend an element \( a_i \) of \( A \)

To affirm \( \langle n, (a_i)_{n}^{n+1-i} \rangle = \langle n', (a'_{i})_{n'}^{n'-1} \rangle \),
- find \( n = n' \) and affirm \( a_i = a'_i \) for each \( 0 \leq i \leq n \)

**Note** - If word problem undecidable, then
inequality is not observable in same sense.

**Constructivist**

Unique choice \( \Rightarrow \) functions interpreted the same way.
Classifying categories as spaces

- Work in opposite of cat $\mathcal{C}$
- Initial $\mathcal{C}$-cat becomes final. Write it as $1$
- Point $\mathcal{1} \rightarrow \mathcal{C}_T \simeq \text{model of } T \text{ in } \mathcal{C}_\text{init}$
- Generalized point $\mathcal{C} \rightarrow \mathcal{C}_T \simeq \cdots \mathcal{C}$
  - Points of $\mathcal{C}_T = \text{models of } T \cdots \mathcal{C}_T = \text{"space of } T\text{-models"}.$
- $f : \mathcal{C}_T_1 \rightarrow \mathcal{C}_T_2$
  - $f$ transforms points $M \rightarrow f M$ ($M : \mathcal{C} \rightarrow \mathcal{C}_T_1$).
  - $f$ transforms models, $f = \text{model of } T_2 \text{ in } \mathcal{C}_T_1 \rightarrow f(M)$ constructed by $\mathcal{C}$-constructions out of $M$.
  - $M$ as functor preserves $\mathcal{C}$-constructions $f(M)$ made by same construction.

Classifying toposes

- Geometric logic
  - Logic $\mathcal{L}$ interpreted using categorical structures $\mathcal{C}$.
  - Given a theory $T$:
    - For $\mathcal{C}$-cat $\mathcal{C}$: category $\text{Mod}_\mathcal{C}(T)$ of models in $\mathcal{C}$
    - For $\mathcal{C}$-functor $\mathcal{C} \rightarrow \mathcal{D}$, base functor $\mathcal{F}_{\mathcal{C} \rightarrow \mathcal{D}} : \text{Mod}_\mathcal{C}(T) \rightarrow \text{Mod}_\mathcal{D}(T)$.
    - Classifying category $\mathcal{C}_T$, with generic $T$-model $\mathcal{M}_T$.
      - $\mathcal{C}$-cat $(\mathcal{C}_T, \mathcal{C}) \rightarrow \text{Mod}_\mathcal{C}(T)$, $\mathcal{F} \rightarrow \text{Mod}_\mathcal{D}(T)$.
    - Idea: $\mathcal{C}_T$ freely generated as $\mathcal{C}$-cat by $\mathcal{M}_T$.
    - Trivial theory, $\mathcal{M}_T$ classified by initial $\mathcal{C}$-cat.
    - $\mathcal{C}$-functor $\mathcal{C}_T \rightarrow \mathcal{C}_T' \simeq \text{model of } T_1 \text{ in } \mathcal{C}_T'$.
Geometric morphisms \( f : \mathcal{E} \to \mathcal{F} \) is 

\[ f^\ast : \mathcal{E} \to \mathcal{F} \text{ preserving colimits, finite limits} \sim \mathcal{E} \leftarrow f^\ast : \mathcal{F} \text{ preserving finite limits} \]

Classifying categories as spaces
- Work in opposite of Cat: \( \mathbf{Cat}^\text{op} \)
- With \( \mathbf{Cat}^\text{op} \) being dual to \( \mathbf{Cat} \)
- Points: \( \mathbb{1} \to \mathcal{T} \) is model of \( \mathcal{T} \) in \( \mathbf{Cat}^\text{op} \)
- Points of \( \mathcal{C} \) are models of \( \mathcal{C} \)
- \( \mathcal{C} \)-Space: \( \mathcal{C} \)-Space:
  - geometric morphism \( \mathcal{C} \rightarrow \mathcal{C} \)
  - \( \mathcal{C} \)-Space

Topos = “space of models” for some geometric theory
Write \( \mathbb{T} \) instead of \( \mathbf{Set}[\mathbb{T}] \) in dual category

Geometric morphism = model transformer

\[ \text{To define } f : \mathbb{T}_1 \to \mathbb{T}_2 \]

Let \( x \) be a model of \( \mathbb{T}_1 \).
Then \( f(x) = \vdots \)

is a model of \( \mathbb{T}_2 \)

NB No problem if \( \mathbb{T}_1 \) has insufficient models in Set
Geometric construction works uniformly for all models in all toposes - including generic model

Geometricity = continuity

For propositional geometric logic:
- same trick works giving frames & frame homomorphisms
  - localic analogue of continuous maps
Propositional theory \( \mathbb{T} \) \( \to \) frame \( \mathbb{S}[\mathbb{T}] \)

Theorem: \( \mathbb{S}[\mathbb{T}] \) = topos of sheaves over \( \mathbb{S}[\mathbb{T}] \)

Geometric morphism \( \mathbb{S}[\mathbb{T}] \to \mathbb{T}_2 \)

\& locale map \( \mathbb{S}[\mathbb{T}] \to \mathbb{T}_2 \)

Geometricity = continuity

More generally: take "(continuous) maps" to be geometric constructions/morphisms
- e.g. sheaves \( \mathbb{T}_{ob} : \) one sort, nothing else
  - model = object
  - geometric morphism \( \mathbb{T} \to \mathbb{T}_{ob} \)
  - object \& \( \mathbb{S}[\mathbb{T}] \)
  - sheaf
  - geometric construction
M \& \( \mathbb{T} \to \) set
  - stalk at \( M \)
Sheaf = "continuous set valued map - map: space of models of \( \mathbb{T} \to \text{space of sets} " \)
Locale constructions

If construction on frames is topos-valid:
also gives construction on bundles

\[ [T,] \rightarrow \text{locale in } 8\{T\} \rightarrow \text{new locale in } 8\{T\} \approx \rightarrow \]
\[ [T,] \]

Geometric locale constructions

Internal frame has internal presentation

\[ \Omega \rightarrow [T] \]

If construction can be done geometrically on presentations

Then bundle construction preserved by pullbacks

\[ \Rightarrow \text{works fibrewise} \]

Fibres are pullbacks along global points

Examples

Powerlocales (localichyrespaces)

Valuation locales

Get geometric characterization of e.g. compactness

UFPs:
- Geometric types
- Ontology = observational
- Geometricity = continuity

\[ \Rightarrow \text{Fibrewise topology (bundles)} \]
Geometricity in general

- Locale construction is geometric if (on bundles) preserved by pullback
- Generalizes geometricity for set constructions (bundle = local homeomorphism)
- Logically: define bundle $p : \mathbb{T}_1 \to \mathbb{T}_2$ by

\[ p(x) = \vdots \]

is a locale

Selected bibliography

Johnstone: *Sketches of an elephant*, vol 2

Joyal & Tierney: *An extension of the Galois theory of Grothendieck* Bundles

Vickers: *Topology via logic* Ontology

Issues of logic, algebra & topology in ontology

(Chapter in "Theory & Applications of Ontology" vol 2)

Locales & toposes as spaces

(Chapter in "Handbook of Spatial Logics"

Topical categories of domains

The double power/locale and exponentiation bundles