Surviving without function types:

Life in an arithmetic universe

Talk given 22 Jun 2010
CT2010 Genova

Steve Vickers
School of Computer Science
University of Birmingham
Work in progress joint with Milly Maietti
- Arithmetic universes as generalized spaces
- An induction principle
- Classical logic – of

Part I
Arithmetic universes

Arithmetic universe

Originated: Joyal (1970s) unpublished
- constructed examples (inc. initial one)
- applied to Gödel’s theorem
General definition?
Pretopos + internal free algebras

Maietti:
Au = list-arithmetic pretopos
pretopos + parametrized list objects

A category is a pretopos if
- has finite limits
- has finite coproducts
- and they are stable and disjoint
- has stable, effective quotients of equivalence relations

cf. Giraud’s theorem
Similar conditions + all small coproducts
+ size constraint

⇒ Grothendieck topos

- has, hence free algebras
- cartesian closed
- power objects
Arithmetic universe = list-arithmetic pretopos
For every A : list (A) has
\[ 1 \xrightarrow{1} \text{List}(A) \xrightarrow{\text{cons}} \text{List}(A) \]
\[ A \times \text{List}(A) \xrightarrow{\text{cons} \times B} \text{List}(A) \times B \]
\[ \langle B, [], ! \rangle \xrightarrow{\text{rec}(y_0)} \text{List}(A) \times B \]
\[ B \xrightarrow{y_0} A \xrightarrow{y} \]
\[ \forall y_0 \exists a! \text{rec}(y_0, a) \]
\[ \text{write } g(a, a) = a \cdot y \]. Then rec(y_0, a) \( \langle a_1, a_n \rangle, b \) = a_1 \ldots a_n \cdot y_0(b)

Classifying AUs
Theory of AUs is cartesian
\[ \Rightarrow \text{can present AUs by generators & relations} \]
Geometric theory
\[ \downarrow \]
AU presentation
Internal types + \[ \exists \text{V} \]
AU stands in for classifying topos
AUs support limited fragment of sheaf theory

Arithmetic spaces
Generalized space = Grothendieck topos
Continuous Map = functor (backwards)
\[ \text{agometric} \]
\[ \text{preserving all colimit} \Rightarrow \text{has right adjoint} \]
\[ \text{also free algebras} \]
Idea: Arithmetic space \( X \) given by \( \text{AU} \times X \)
Map \( f : X \rightarrow Y \) is functor \( \text{AU} \times X \leftarrow \text{AU} \times Y \)
\[ \text{preserving finite colimit} \]
\[ \text{finite limit} \]
List

Part II
INDUCTION
**Induction I**

\[ \phi \in \mathbb{N}, \text{ Suppose: } \phi(0) \text{ and } \forall n. (\phi(n) \rightarrow \phi(n+1)) \]

\[ \Rightarrow \forall n. \phi(n) \]

**Induction for \( \phi \rightarrow \psi \)**

Two predicates \( \phi, \psi \) on \( \mathbb{N} \)

\[ \phi, \psi \in \mathbb{N} \]

Want \( \forall n. \phi(n) \rightarrow \psi(n) \)

Function types give predicate \( \phi \rightarrow \psi \)

Use same induction.

**Base case** \( \phi(0) \rightarrow \psi(0) \)

**Induction step** \( \forall n. (\phi(n) \rightarrow \psi(n)) \Rightarrow (\phi(n+1) \rightarrow \psi(n+1)) \)

\[ \Rightarrow \forall n. \phi(n) \rightarrow \psi(n) \]

**Without function types?**

\[ \text{classical logic} \]

\[ \phi(n) \rightarrow \psi(n) \equiv \neg \phi(n) \lor \psi(n) \]

\[ (\phi(n) \rightarrow \psi(n)) \Rightarrow (\phi(n+1) \rightarrow \psi(n+1)) \]

\[ \equiv \neg \phi(n) \lor \psi(n) \Rightarrow \neg \phi(n+1) \lor \psi(n+1) \]

\[ \equiv \neg \phi(n) \Rightarrow (\neg \phi(n+1) \lor \psi(n+1)) \]

\[ \neg \phi(n) \lor \psi(n) \Rightarrow (\neg \phi(n+1) \lor \psi(n+1)) \]

\[ \equiv \phi(n+1) \rightarrow \phi(n) \lor \psi(n+1) \]

\[ \land \phi(n+1) \land \psi(n) \rightarrow \psi(n+1) \]

**Theorem (Maieth Nickels)**

In any arithmetic universe: if have \( \phi, \psi \in \mathbb{N} \)

\[ \phi(0) \rightarrow \psi(0) \]

\[ \forall n. (\phi(n+1) \rightarrow \phi(n) \lor \psi(n+1)) \]

\[ \forall n. (\phi(n+1) \lor \psi(n) \rightarrow \psi(n+1)) \]

Then \( \forall n. (\phi(n) \rightarrow \psi(n)) \)
Proof

Define \( A(k) \subset N \)
\[ A(k) = \{ j \in N \mid j \leq k, \; \phi(j), ..., \phi(k) \} \]

Define \( f_k : A(k) \rightarrow \psi(k) \) recursively so \( f_k(j) = k \)
\[ f_k(0) = \psi(0) \text{ base case} \]
\[ f_k(j) = \begin{cases} f_k(j-1) & \text{ if } j > 0 \\ k \land \psi(j) & \text{ if } j = 0 \end{cases} \]

If \( \phi(k) \) then \( f_k(k) = k \) - use \( f_k(k) \)

Recursion variant = \( k \land \psi(k) \)

\( j < k \) : From \( f_{k-1}(j) \) get \( \psi(k-1) \)
\[ \therefore \phi(k) \land \psi(k-1) \text{ so } \psi(k) \text{ by IS2} \]

Summary

If classical logic in \( A \) then induction proof for \( \phi(n) \rightarrow \psi(n) \)

Reduces to new induction principle

Part III

Classical logic of subspaces

- Replace subsets of \( N \) by subspaces
- Open subspaces \( \Rightarrow \) Boolean complement
- Boolean calculations on subspaces \( \Rightarrow \) valid properties of subsets

Subspace triple

X an AS: Subspace triple = \( u_i \uparrow \) two subsheaves

Sheaf = \( \phi(x) \) object of AX

Corresponding subspace is \( X[u_i \leq V] \)

Preorder \( u_i \leq V \iff AX[u_i \leq V] \text{ also has } u_i \leq V' \)

Meets \( u_i \land u_j \leq 0 \)

Special cases for \( \phi, \psi \rightarrow 1 \)

Open \( \psi = \frac{1}{u_i} \phi \leq \psi \)
Closed \( X - \phi = \phi \leq (X-\phi) \land \psi \)

Crescent \( (X-\phi) \land \psi \)

Cod crescent \( (X-\phi) \lor \psi \)
Representation theorems

1. Local homeomorphisms
   \[ AX[a:1 \to A] \simeq AX/A \]
   Adjoin generic element of A
   \[ A \Delta AX \]

Special case: opens
   \[ A \phi \iff 1 \]
   \[ A \phi = AX/\phi \]

Corollary
   \[ \psi \leq u \leq V \]
   \[ \psi \leq u \leq V \iff u \leq V \phi I \phi \]

Coresent
   \[ (X-\phi) \wedge \psi \leq u \leq V \]
   \[ (X-\phi) \wedge \psi \leq u \leq V \]

Coresent
   \[ \phi \leq u \leq V \]
   \[ \phi \leq u \leq V \]

Corollaries
   \[ (X-\phi) \wedge (X-\phi) = X- \phi \]

Subspace for subobject join \( \psi \vee \phi \)

\[ \psi \vee \phi \]

\[ \psi \leq (X-\phi) \vee \phi \]

\[ \phi \leq (X-\phi) \wedge \phi \]

\[ (X-\phi) \wedge \phi = 0 \]

Closed subspace of 1 is Stone

X-\phi \quad BA of clopns

\[ \phi \leq 2/(0=1 \text{ if } \phi) \]

Sheaves F over \( B\phi \)

\[ F(0) = 1 \]

\[ T \phi \rightarrow T (\phi) \quad \text{coequalizer} \]

Thm: \( sh(B\phi) \simeq Alg(T \phi) \simeq A(X-\phi) \)

Reflective subset of \( AX \)
Lattice structure

Lemma

1. \( \bigvee \{ Y \cap (X_{i} \cup \phi_{i}) \mid i \in I \} \) exists and equals
2. \( \bigwedge \{ X \cap \bigvee_{i \in I} \phi_{i} \mid \{ i, ..., n \} = \emptyset \} \)

Proof

1. Conjunctions in (1) are upper bounds for disjunctions in (2).
2. Over (1), suppose \( \phi_{i} : Y \cup (X_{i} \cup \phi_{i}) \cup \psi_{i} \leq \psi_{j} \)
   adjoin all \( \phi_{i} = \psi_{k} \). Prove by induction \( n = 1 \)
   \( u \cup \bigwedge_{i \in I} \phi_{i} = \bigvee \)

Corollary

1. Finite joins of crescents exist and are meets of cocrescents.
2. Any \( Y \) distributes over those joins.

Induction

\[ \phi, \psi \rightarrow N \text{ in } \mathbb{A}^X \]
\[ \phi(n), \psi(n) \rightarrow 1 \text{ in } \mathbb{A}^X[n : 1 \rightarrow N] \]
\[ \phi(n+1), \psi(n+1) \]

Theorem

Suppose \( \phi(0) \leq \psi(0) \) in \( \mathbb{A}^X \)
\[ \phi(n), \psi(n) \leq \phi(n+1), \psi(n+1) \] over \[ \mathbb{A}^X[n] \]

Then \( \phi = \psi \) in \( \mathbb{A}^X \)

Proof

\[ (X[n] - \phi(n)) \cup \psi(n) \leq (X[n] - \phi(n)) \cup \psi(n+1) \]
\[ i.e., \phi(n+1) \leq \phi(n) \cup \psi(n+1) \]
\[ \phi(n+1) \cap \psi(n) \leq \psi(n+1) \]

From \( \mathbb{A}^X[n] \approx \mathbb{A}^X/N \) deduce corresponding relations in \( \mathbb{A}^X \).

Conclusions

Classical logic of (some) subspaces
even when logic of subobject not classical
\( \Rightarrow \) can work with implications
even when no internal exponentials

General moral: Good properties of spaces spoiled
when you discretize (take set of points)
eg. Closed complement properly a Stone space
- don’t expect a subobject of 1