

# Applications of geometric logic to topos approaches to quantum theory

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## Abstract

This document describes my 3-year project “Applications of geometric logic to topos approaches to quantum theory”, to start in 2009 with funding from the UK Engineering and Physical Sciences Research Council (EPSRC) for a post-doctoral Research Assistant and a PhD studentship.

After an overview of the background and programme of work, it leads on to a description of the high-grade postdoctoral post funded as part of the project.

## 1 Summary

- The so-called “geometric logic” (with its associated type theory) makes it easier to reason with point-free topology in toposes.
- I have a long-term programme of testing out the scope of applicability of geometric logic in working mathematics.
- Two groups of researchers (Imperial, Nijmegen) have identified the potential of formulating quantum physics internally in toposes.
- The Nijmegen group has already identified the usefulness of geometric reasoning, but there are gaps in current knowledge of the relevant point-free theory of  $C^*$ -algebras.
- The project is to develop geometric logic in those directions, as a case study that is not only highly substantial in its own logical terms but also promises to lend considerable clarification to the physicists’ own engagement with toposes.
- The project will also examine the connections with other categorical accounts, such as the use of monoidal categories for quantum protocols at the Oxford group of Abramsky and Coecke.

## 2 Background

### 2.1 Point-free topology and geometric logic

*Geometric logic* (I shall consistently use the word “geometric” in this logical sense) arose in topos theory and dates back at least to [MR77]. It has a particular use in dealing with the non-constructivity of topology in its usual “point-set”

form, and this use can be seen explicitly in [Wra79] and implicitly in much other topos theory. It has also been used in computer science as a “logic of observations” of computer programs – see e.g. [Abr91], [Vic89]. As explained further below (Section 2.2), I have made it a central part of my own work, key examples being [Vic99] and [Vic04a] as geometrization studies inspired by [Abr91] and [Hyl81].

The underlying problem is that, for the internal mathematics of a topos, topology needs to be done in the “point-free” form of locales (or of toposes, as Grothendieck’s generalized topological spaces): otherwise important results fail to hold. However, the point-free formulation can itself be a barrier to mathematical work, first by obfuscating the exposition in lattice theory and second by the pervasive phenomenon of non-spatiality – even if one tried to use points, there are too few of them.

A particularly important fact in [JT84] (also in [FS79]) is that if the topos is that of sheaves over  $X$  then its internal locales are equivalent to (external) locale maps to  $X$ , in other words localic bundles over  $X$ ; the sheaves themselves then correspond to bundles that are local homeomorphisms. The essential import of the word *bundle* is that the fibres are to be thought of as a variable space, parametrized by base point. Note that we do not assume the map is surjective (so some fibres may be empty).

In full generality, *geometric reasoning* (in this logical sense) is characterized as that which is preserved by pullback of the bundles along geometric morphisms and hence remains valid fibrewise. For the local homeomorphisms it is preserved by inverse image functors. The notion is particularly important in connection with *geometric theories*, used to present locales and hence (in the internal setting) bundles. Internal models of the theory are equivalent to sections of the bundle, and moreover the passage from theory to bundle is geometric. It can be deduced that a geometric transformation of models of one theory into models of another gives a map of bundles, and this effectively restores the points to point-free spaces: a space is the space of models of a geometric theory, and a map is a geometric transformation of models. (To put it another way, geometricity gives access to the *generalized* points – maps with that space as codomain.) Thus, *so long as the mathematical reasoning respects geometricity*, internal locale theory can be conducted fibrewise (for the bundles) and using points. This greatly aids the transparency of the work.

The geometrization proviso is wholly non-trivial. Much of the internal mathematics of toposes is non-geometric. My own work has geometrized important pieces of mathematics, generally by careful consideration of whether a collection should be a set (with naturally discrete topology) or a space (with some natural non-discrete topology), and by developing tools of locale theory including powerlocales. Thus the geometrization programme, despite its foundational advantages, remains to be tested out in developing practical mathematics.

Geometric logic arose in topos theory, and the machinery of toposes makes precise the notion that geometric transformations of points are equivalent to continuous maps (= geometric morphisms) between toposes. However, geometric mathematics also has predicative qualities, in that the archetypal impredicative construction, the powerset, is not geometric. In the predicative treatment of formal topology geometric transformations can be converted case-by-case into continuous maps, though as yet there is no general metatheorem for this. Thus geometricity in some form also promises to be a unifying principle for different

foundational settings.

## 2.2 My own work

Already since my 1989 book [Vic89] I have been known for my work on point-free topology, but since around 1992 (with [Vic92]) I have been focusing on geometric logic and its relationship with point-free topology, with particular regard to two aspects.

1. Its ability to provide a point-based exposition for developing point-free topology (including topology in its generalized sense, namely the topos as generalized topological space).
2. Formalizing a geometric type theory that eliminates infinitary disjunctions in favour of existential quantification over geometric type constructs.

Aspect (1) has long been an implicit part of topos-theoretic folklore, but I have described it very explicitly in a series of works (in most detail in [Vic99], [Vic04a], [Vic07a]). It is in itself a major long-term programme, since it raises the non-trivial question of how much standard mathematics can be conducted within the geometricity constraints. I have given positive answers in case studies in domain theory ([Vic99], [Vic01]), process theory ([RV03]), quasimetric spaces ([Vic05a], [Vic03]) and real analysis ([Vic08], [Vic09a]). [Vic99] and [Vic05a] exemplify the way geometric reasoning can enable constructive point-free topology (including toposes) to resemble a more standard treatment. A major technical tool is the use of powerlocales (point-free hyperspaces). I have proved the geometricity of the powerlocale constructions in general ([Vic04a]), and, for a variety of classes of spaces, given a convenient geometric construction of their powerlocales ([Vic93], [Vic04b], [Vic03], [Vic09a]). The resulting geometric hyperspace techniques have been used in a variety of applications, including the Heine-Borel Theorem ([Vic03]), the Intermediate Value Theorem and Rolle's Theorem ([Vic09a]), and valuations and integration ([Vic08]).

Aspect (2) is less well developed, but carries the important prospect of giving a finitary formalism for a substantial fragment of geometric logic, adequate for conducting important mathematics. Moreover, I conjecture that this would apply in foundational settings other than topos theory, such as predicative type theory. The topos-theoretic proofs of metatheorems justifying geometric point-based reasoning are impredicative, but the geometric type theory itself is predicative and I have been exposing the techniques to predicative foundations ([Vic06], [Vic05b], [Vic07b], [Vic09b]). In addition, with Palmgren ([PV07]) I have given a formulation and predicative proof of the initial algebra theorem for cartesian theories, an important construction in geometric logic. We described applications to the construction of classifying categories with a view to using these as predicative analogues of the classifying toposes, which are a key construction in proving the topos-theoretic metatheorems on point-based geometric reasoning.

Further details, preprint versions of papers, and slides from talks can be found on my website at <http://www.cs.bham.ac.uk/~sjv>. The talks "Locales via Bundles" and "The Topos Approach in the Qubit Case" are particularly relevant.

### 2.3 Quantum physics and geometricity

Classical physics takes a “realist” approach in which the values of all physical quantities are determined by a pure state of the physical system. Hence a physical quantity can be represented mathematically as a map from pure states to values. In quantum physics, the Kochen-Specker Theorem says this is impossible. There, a physical quantity is represented as a linear operator on a Hilbert space  $H$ . Under an instrumentalist interpretation of quantum theory (though clearly this would be questionable for any kind of quantum cosmology), a measurement of that quantity can take its values in a range, the spectrum of the operator, comprising in the finite dimensional case the eigenvalues of the operator. The state (element of  $H$ ) is then not deterministic, and determines only the probabilities with which those spectral values may be the result of the actual measurement. For an operator with a continuous spectrum, the state gives a probability distribution on the spectrum.

In a sense, the probabilistic aspect is not the heart of the quantum problem: for even in classical systems one can consider mixed, probabilistic states based on incomplete knowledge. The problem is whether there are deterministic (classically pure) states on which the probabilistic states can be understood as probability distributions (mixed states). For a single operator, the answer is Yes – the deterministic states are given by the spectrum. This extends to any set of pairwise commuting operators, since they are simultaneously diagonalizable. This situation is mathematically consistent with the existence of “hidden state variables” that determine the outcome of measurements. The problem comes with operators that do not commute. Their values cannot be measured simultaneously, and an example of this is the Heisenberg uncertainty principle. The Kochen-Specker Theorem shows that there cannot be a system of classically pure states in terms of which the quantum pure states (elements of the Hilbert space) are mixed states.

[IB98] initiated a “neo-realist” approach, which aims to restore a realist logic to quantum physics by interpreting it in the internal logic of a suitable topos. In effect it looks at all possible ways of using commutative subalgebras of operators to gain realist notions of state – each commutative subalgebra has its own Gelfand-Naimark spectrum. This general idea has also been adopted by the Nijmegen group in [HLS08b], in a style that declaredly (see [HLS08a]) respects geometricity.

Given a non-commutative algebra  $A$ , one can construct a space  $C(A)$  of commutative subalgebras. Over this there is a “Gelfand-Naimark” bundle, the fibre over each subalgebra being its spectrum. Classically pure states would be global sections of this and Kochen-Specker (as interpreted in [IB98]) says there are none. But if we construct, fibrewise, the spaces of probability distributions, then we find that the resulting distribution bundle *does* have global sections. (More correctly, [HLS08b] construct the space of probability *valuations*, defined only on open subsets and not arbitrary measurable subsets.) These correspond to quantum mixed states and include, as their extreme points, the quantum pure states. The technicalities of the fibrewise constructions are taken care of by working internally in a topos derived from the subalgebra space  $C(A)$ . (A current open question, which we hope to throw light on, is how to express superposition, as addition of pure states, in the internal mathematics.)

In [DI08], the topos is that of presheaves over  $C(A)$ , and the subalge-

bras themselves are von Neumann subalgebras. Roughly speaking, the use of presheaves is forced by their point-set approach to the spectrum: for the spectrum to be an object in the topos, it must depend contravariantly on subalgebra inclusion. Also, their use of von Neumann subalgebras may be seen as being measure-theoretic in flavour rather than topological. By contrast, in [HLS08b] the topos is that of covariant functors from  $C(A)$  to **Set**, while the subalgebras used are  $C^*$ -subalgebras. Their point-free approach (in accord with geometricity) allows them to use the topos of covariant functors, which also has the advantage of allowing an internal commutative  $C^*$ -algebra. Without prejudice to any physical arguments, the present project will largely consider it rather than [DI08]. However, there are openings for exploring other toposes by bringing in topology more fully. For example, for the qubit case  $A = M_2(\mathbb{C})$  the non-trivial commutative subalgebras form a smooth manifold, the real projective plane, and the spectral bundle is the double cover of that by the 2-sphere. This corresponds to using the topos of sheaves over the projective plane.

## 2.4 Constructive Gelfand-Naimark spectrum

The Gelfand-Naimark Theorem for commutative  $C^*$ -algebras (both complex and real) establishes a duality between them and compact Hausdorff spaces. For the space  $X$ , the algebra is the set of continuous maps from  $X$  to  $\mathbb{C}$  or  $\mathbb{R}$ . In [HLS08b] it is important to be able to apply this in the internal mathematics of a topos, and this was shown possible in [BM06]: there is a duality, topos-valid, between  $C^*$ -algebras and compact, completely regular locales.

A question raised in [BM06] is whether this can be tightened to give a point-free treatment of the  $C^*$ -algebra as well as the compact Hausdorff space. This is important because the algebra is a vector space over  $\mathbb{C}$  or  $\mathbb{R}$ , and so in principal suffers from the same constructive problems in a point-set treatment as do  $\mathbb{C}$  and  $\mathbb{R}$  themselves. At the same time it would be good to seek a way to relax the condition of complete regularity to mere regularity.

The algebraic and norm structure of the  $C^*$ -algebra structure can be readily expressed in the category of locales. What is difficult is the condition that it is complete with respect to the norm. Existing results [Vic05a] show how to construct a *locale* that is the completion of a generalized metric *space*. That allows hope that completeness of the localic  $C^*$ -algebra could be expressed by saying that it arises in this way. However, there are reasons to suspect that the corresponding compact regular locale might in fact still be completely regular. To cover all compact regular might require a proper account of generalized metric locales and their completeness – which would be a good thing in itself.

$C^*$ -algebras as such are not the only relevant structure. The theory of Riesz spaces and the Riesz Representation Theorem are also relevant. Constructive results on these [CS08] have already been needed for the programme of [HLS08b]: this is because states correspond to integrals, which in turn correspond to valuations (in the sense of measure-like functions defined on the lattice of open sets).

## 2.5 Current activity

Though its roots date back to 1998, the neo-realist programme has come to fruition in its current form only recently with the Döring-Isham collaboration

[DI08], yet it has already had sufficient impact to reach [Smo06] and [Mat07], the first mentions of topos theory in the *New Scientist*. Meanwhile, the Nijmegen work [HLS08b] has also come out.

Over that period a series of workshops at Imperial on categories in theoretical physics have now developed into the very active Imperial-Oxford workshop series on *Categories, Logic and Foundations of Physics* (CLOP; <http://categorieslogicphysics.wikidot.com>). This covers a wide range of interactions between physics and category theory, including not only the topos approaches but also the monoidal categories as used by the Oxford group of Abramsky and Coecke [AC07] in their study of quantum protocols. It provides an excellent forum for comparing the different categorical structures that are arising.

## 3 Programme and Methodology

### 3.1 Overall aims

The hope behind the topos approaches is that, as a radical re-understanding of the formulation of quantum theory, they may cast light on the fundamentally important problem of quantum gravity. The present project does not pretend to address those deep issues of physics. Rather, it seeks to respond to those investigations in physics by clarifying what is achieved by the topos approach, exploiting known techniques of topos theory to translate “topos-internal” to more familiar concepts such as “fibrewise”.

Specifically, the aim is to investigate the applicability of techniques of geometric logic with particular regard to the topos approach to quantum theory, including [DI08] but especially the geometric treatment announced in [HLS08a] and initiated in [HLS08b].

Our aim is to gain the expositional advantages of geometric logic and develop a more transparent “pointwise” style of exposition for the geometric treatment of non-commutative  $C^*$ -algebras and quantum issues. Ideally, physicist readers should be able to read the mathematics in a “fibrewise” way without having to calculate sheaf representations of the internal mathematics. Essentially this is because geometricity allows internal locales to be understood as bundles. Thus we hope to make the programme more accessible to physicists for whom the machinery of topos theory is too much of a prerequisite. However, the work is also expected to lead to new developments in how the toposes are used to treat the physics, and clarification of the relationship with other categorical approaches such as that of [AC07].

At the same time, it will provide a substantial and significant case study to test out the scope of application of geometricity techniques. It thus forms a natural step in the geometrization programme, of seeking to express as much of mathematics as possible in a form that respects geometricity. It is also expected to develop the underlying metatheory of geometricity, with benefits not only to topos theorists but also to constructive mathematicians in different foundational settings. In addition, the physics setting is provided new examples and calculations of topos-internal  $C^*$ -algebras and their spectra.

The success of the work will be judged by its publication of (i) geometrization of substantial mathematical results hitherto established only to classical or

intuitionistic validity; (ii) mathematical exposition of topos-theoretic quantum foundations using geometricity to civilize the topos machinery and aid explicit computation of examples; (iii) metatheorems improving the applicability of geometric reasoning, in topos theory and in other foundational settings.

## 3.2 Methodology

The spine of the work will be to follow a core technical geometrization question for the topic: How does one make Gelfand-Naimark duality fully point-free? Initially, this will also form a convenient focus for establishing the project by bringing together and developing previous work on generalized metric spaces, Riesz spaces and integration. (See Packages 1 and 2 under Section 3.3.) The understanding accumulated in this spine will underpin derived work in two directions. The first direction (Package 3) is towards the quantum formalism, investigating mathematical structures and exposition for a geometrically topos-theoretic account of quantum foundations, with case studies as appropriate. The second direction (Package 4) is towards constructive mathematics including metamathematical investigations of geometric logic.

The work will draw on and develop existing connections with the groups at Imperial, Nijmegen and Oxford.

## 3.3 Programme of work

The specific programme of work divides into four Work Packages as follows.

1. A fully point-free account of Gelfand-Naimark duality for commutative  $C^*$ -algebras. This would include identifying the correct notions of localic  $C^*$ -algebra. It is expected that the localic techniques of completion for generalized metric spaces (see [Vic05a]) will be applicable here, as well as powerlocale techniques, and will themselves gain further development. Related relevant topics, including Riesz spaces, the Riesz Representation Theorem, point-free accounts of integration and the Peter-Weyl Theorem, will also be investigated.
2. An appropriate account of non-commutative localic  $C^*$ -algebra, and suitable toposes to classify the commutative subalgebras and provide the basis for the geometric development. (These would not necessarily be presheaf or functor toposes as in the existing approaches. Even in the qubit case there is topology on the collection on commutative subalgebras - the non-trivial ones form a manifold homeomorphic to the projective plane. This suggests using a topos of sheaves over that space.) Suitable toposes for composite quantum systems, adequate for capturing phenomena of entanglement.
3. Development of pointwise and fibrewise styles of exposition for the geometricity programme, with reference to fibre bundles; application to specific quantum systems including that of qubits and the relationship with the monoidal categories of [AC07].
4. Metamathematical results allowing the pointwise reasoning to be interpreted in foundational settings other than topos theory, in particular predicative type theory. At present there is good topos-theoretic content to an

assertion that *geometricity is continuity* (or, more specifically, “geometric transformations = continuous maps”). In settings other than topos theory at present one can only verify this case by case.

As mentioned under Section 3.2, the first two will be conducted consecutively as the spine of the project. The other two will be conducted in parallel to exploit understanding accumulated in the first two.

## 4 The research assistantship

The project has been awarded 3 years’ funding for a postdoctoral research assistant, to start between May and October 2009. The funding provides for appointment at a high salary grade for a researcher with sufficient qualifications and experience in the relevant fields. The essential fields of expertise are those of toposes, constructive mathematics and point-free topology, and for appointment at the high grade the researcher would need to have established already an excellent track record in those fields. Some familiarity with quantum theory and quantum computing will also be needed.

Apart from helping to drive the research programme outlined above, the researcher will be expected to participate in supervising a PhD student funded by the project and to spend up to 6 hours a week helping to deliver taught courses in the School of Computer Science.

### 4.1 Working at Birmingham

Birmingham is the second largest city in the UK, centrally placed with good rail and air connections to the rest of the country and to Europe.

Though the pure mathematical core (topos theory) of this interdisciplinary project is to be applied with physics in view, the topic has also been much investigated for its interaction with Theoretical Computer Science and the project will be conducted in the School of Computer Science at the University of Birmingham.

The School is ranked equal 7th amongst UK computer science departments in the proportion of its research activity awarded 4\* (the highest possible level).

It has a strong, coherent group in Foundations of Computer Science and Theory of Programming Languages, with regular seminars and lunch meetings, and lively discussion of current work. Straddling the interface between computer science and pure mathematics, the group brings both mathematical skills to computer science applications and computational insights to mathematical questions and embodies much expertise relevant to this project including constructivity and formalization. Apart from myself, members of the group include Dr Martín Escardó, whose work on exact real computation and its semantics and reasoning principles frequently uses locales and constructive reasoning about them; Prof. Achim Jung, who has an interest in locales, and expertise in the probabilistic powerdomain related to the valuation locale; Dr Paul Levy and Dr Dan Ghica.

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