Exercises 2

Toposes

These exercises needed more detail than I expected. The essential message is that a sheaf on $\text{Idl } P$, considered as a continuous map from $\text{Idl } P$ to $\text{Sets}$, has to take directed joins to directed colimits, and so is defined by a functor from $P$ to $\text{Sets}$.

1. Let $p : Y \to X$ be a local homeomorphism, let $x \subseteq x'$ be points of $X$ and let $y \in p^{-1}(\{x\})$. Show that there is a unique $y' \in p^{-1}(\{x\})$ such that $y \subseteq y'$. Deduce that $\text{Fib} : X \to \text{Sets}$, defined by $\text{Fib}(x) = p^{-1}(\{x\})$, is functorial with respect to the specialization order $\subseteq$.

2. Now suppose $X = \text{Idl } P$ with its Scott topology, where $(P, \leq)$ is a poset. Again let $p : Y \to X$ be a local homeomorphism. Show the following.

(a) Suppose $a \in Y$ and $V$ is an open neighbourhood of it. Show that there is some $y \subseteq a$ such that $y \in V$ and $p(y)$ is a principal ideal $\downarrow x$.

(b) If $y \in Y$, let us write $\sqsupseteq y = \{y' \mid y \subseteq y'\}$. Show that if $p(y)$ is principal then $\sqsupseteq y$ is open, and every open is a union of such opens $\sqsupseteq y$.

(c) If $I \in \text{Idl } P$ and $a \in \text{Fib}(I)$, show that there are some $x \in I$ and $y \in \text{Fib}(\downarrow x)$ such that $a = \text{Fib}(\downarrow x \subseteq I)(y)$.

(d) Suppose $I \in \text{Idl } P$, $x_i \in I$ and $y_i \in \text{Fib}(x_i)$ ($i = 1, 2$), and $\text{Fib}(\downarrow x_1 \subseteq I)(y_1) = \text{Fib}(\downarrow x_2 \subseteq I)(y_2) = a$ (say). Show that there is some $x \in I$ with $x_i \subseteq x$ and $\text{Fib}(\downarrow x_1 \subseteq \downarrow x)(y_1) = \text{Fib}(\downarrow x_2 \subseteq \downarrow x)(y_2)$.

It follows from (c) and (d) that $\text{Fib}(I) = \text{colim}_{x \in I} \text{Fib}(\downarrow x)$.

3. Let $F : P \to \text{Sets}$ be a (covariant) functor. Define a poset $Q = \{(x, y) \mid y \in F(x)\}$, with $(x, y) \leq (x', y')$ if $x \leq x'$ and $y' = F(x \leq x')(y)$. The monotone projection $(x, y) \mapsto x$ extends to a continuous map $p : \text{Idl } Q \to \text{Idl } P$, $p(J) = \downarrow \{x \mid (x, y) \in J\}$. Show that it is a local homeomorphism. Show that $F$ is naturally isomorphic to $\text{Fib} \circ \downarrow$

4. Let $X = \text{Idl } P$ with its Scott topology, where $(P, \leq)$ is a poset, $p : Y \to X$ be a local homeomorphism. Show that $Y$ is homeomorphic (over $X$) to the local homeomorphism constructed in question (3) from $F$ defined as $F(x) = \text{Fib}(\downarrow x)$.
1. By the local homeomorphism property we can find an open \( U \) for \( Y \) such that \( y \in U \) and \( p \) maps \( U \) homeomorphically to an open \( p(U) \). We find \( x' \in p(U) \) (using \( x \subseteq x' \)), and so there is a unique \( y'_0 \in U \) such that \( p(y'_0) = x' \). Suppose we can find some \( y' \in p^{-1}(\{x\}) \) such that \( y \subseteq y' \). Then it must be in \( U \) (because \( y \subseteq y'_0 \)) and so \( y' = y'_0 \) (regardless of \( U \)). This proves uniqueness of \( y' \). It therefore suffices just to show that \( y \subseteq y'_0 \). Suppose \( V \) is open for \( Y \) and \( y \in V \). Then \( U \cap V \) is open and contains \( y \). Its image \( p(U \cap V) \) must be open in \( X \), and contains \( x \) and hence \( x' \), and it follows that \( y'_0 \in U \cap V \). Functionality of \( \text{Fib} \) follows from the fact that \( \subseteq \) is a preorder.

2. (a) Let \( U \) be an open neighbourhood of \( a \) such that \( p \) maps \( U \) homeomorphically to an open neighbourhood of \( p(a) \). By taking the intersection with \( V \), we can assume that \( U \subseteq V \). Since \( p(U) \) is Scott open, there is some \( x \in p(a) \) such that \( \downarrow x \subseteq U \). Let \( y \) be its preimage in \( U \). From the argument of question (1) we see that \( y \subseteq a \) (and so \( a = \text{Fib}(\downarrow x \subseteq I)(y) \)).

(b) Suppose \( p(y) = \downarrow x \) and note that \( U_y = \{ I \mid x \in I \} \) is open. Apply the proof of part (a) with \( V = p^{-1}(U_y) \). Then \( \sqsubseteq y = U \cap p^{-1}(U_y) \) is open. The rest now follows from part (a).

(c) This follows from part (a), taking \( V = Y \).

(d) Apply part (a) with \( V = (\sqsubseteq y_1) \cap (\sqsubseteq y_2) \).

3. Let \( J \) be an ideal of \( Q \), and choose \( (x, y) \in J \). Then \( U_{xy} = \{ J' \mid (x, y) \in J' \} \) is an open neighbourhood of \( J \). But it is itself an ideal completion, of \( \uparrow (x, y) = \{(x', y') \in Q \mid (x, y) \leq (x', y')\} \). If \( (x, y) \leq (x', y') \) then \( y' \) is uniquely determined by \( x' \), and it follows that the projection \( (x', y') \mapsto x' \) is 1-1 and onto \( \uparrow x \). Hence \( p \) maps \( U_{xy} \cong \text{Idl}(\uparrow (x, y)) \) homeomorphically to \( \text{Idl}(\uparrow x) \).

Suppose \( y \in \text{Fib}(x) \). Then \( p(\uparrow (x, y)) = \downarrow x \), so we get a function \( f_x: \text{Fib}(x) \to \text{Fib}(\downarrow x) \), defined by \( f_x(y) = \downarrow (x, y) \). If \( x \leq x' \) then \( (x, y) \leq (x', y') \) and \( \downarrow (x, y) \subseteq \downarrow (x', y')(y) \). It follows that \( \downarrow (x', y') = \text{Fib}(\downarrow x \subseteq \downarrow x')(y) \) and hence that \( f_x \) is natural in \( x \). We must also show that it is an isomorphism. Suppose \( J \in \text{Fib}(\downarrow x) \), i.e. \( p(J) = \downarrow x \). Note that for any \( (x', y') \in J \) we have \( x' \leq x \). Since \( x \in p(J) \), we have some \( (x', y) \in J \) with \( x \leq x' \), so \( x' = x \) and \( (x, y) \in J \) and \( \downarrow (x, y) \subseteq J \). We show that \( J = \downarrow (x, y) \). If \( (x'', y'') \in J \) then let \( (x''', y''') \in J \) be an upper bound for \( (x'', y'') \) and \( (x, y) \). We must have \( x''' \leq x \) and so \( y''' = y \) and we deduce \( (x'', y'') \leq (x, y) \). It follows that \( f_x \) is onto; clearly it is 1-1.

4. I write \( p' : \text{Idl} Q \to X \) for the second local homeomorphism (constructed from \( F \)). Define \( f : Y \to \text{Idl} Q \) by \( f(y) = \{(x', y') \mid x' \in p(y) \text{ and } y' \subseteq y\} \) and \( y = \text{Fib}(\downarrow x' \subseteq p(y))(y') \). From question (2) we know this is an ideal. Now define \( g : \text{Idl} Q \to Y \). Given \( J \), take any \( (x, y) \in J \) (it doesn’t matter which) and define \( g(J) = \text{Fib}(\downarrow x \subseteq p'(J))(y) \). If and \( g \) are inverse. To show they are continuous, the inverse image \( f^{-1}(U_{xy}) \) is just \( \sqsubseteq y_o \) and hence is open by 2 (b), and conversely.