Exercises 3
Quantum

The three Pauli matrices are defined as follows

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
\[ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \]
\[ \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

We also write 1 for the identity matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

If \( |\phi_i\rangle = \begin{pmatrix} u_i \\ v_i \end{pmatrix} \) are two vectors in \( \mathbb{C}^2 \) then their inner product is

\[ \langle \phi_1 | \phi_2 \rangle = \bar{u}_1 u_2 + \bar{v}_1 v_2. \]

The vectors are orthogonal if \( \langle \phi_1 | \phi_2 \rangle = 0 \), and the length of \( |\phi_i\rangle \) is \( \sqrt{\langle \phi_i | \phi_i \rangle} \). (Note that \( \langle \phi_i | \phi_i \rangle \) is a non-negative real.)

The vectors are orthonormal if they are orthogonal and both of length 1.

Fact: If \( M \) is self-adjoint\(^1\) then there is an orthonormal basis of eigenstates \( |\phi_1\rangle, |\phi_2\rangle \) with respect to which \( M \) has diagonal form

\[ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \]

where \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues.

1. Show that

\[ \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1, \]
\[ \sigma_x \sigma_y = -\sigma_y \sigma_x = i \sigma_z \]
\[ \sigma_y \sigma_z = -\sigma_z \sigma_y = i \sigma_x \]
\[ \sigma_z \sigma_x = -\sigma_x \sigma_z = i \sigma_y \]

2. Show that any \( 2 \times 2 \) complex matrix \( M \) can be written uniquely as a complex linear combination

\[ M = a_0 1 + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z. \]

3. What conditions on the coefficients \( a_0, a_x, \) etc. are equivalent to \( M \) having all of the following conditions? (I shall call these Bloch matrices.)

(a) \( M \) is self-adjoint (Hermitian)?

(b) \( M \) has trace 0? (The trace of \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) is \( \alpha + \delta \).)

(c) \( M^2 = 1 \).

4. If \( M \) is a Bloch matrix, what are its eigenvectors?

5. Consider \( M \mapsto \{ |\psi\rangle \mid M |\psi\rangle = |\psi\rangle \} \), the fix-space of \( M \). Show that this gives a bijection between Bloch matrices and rays of \( \mathbb{C}^2 \).

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\(^1\)more generally: normal
Solutions

1. These are straightforward matrix multiplications.

2. \[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} = \frac{\alpha + \delta}{2} 1 + \frac{\beta + \gamma}{2} \sigma_x + \frac{-\beta + \gamma}{2i} \sigma_y + \frac{\alpha - \delta}{2} \sigma_z
\]

For uniqueness, it suffices to show that if \(a_0 1 + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z = 0\) then all the coefficients are zero. We have

\[
a_0 1 + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z = \begin{pmatrix}
a_0 + a_z & a_x - ia_y \\
a_x + ia_y & a_0 - a_z
\end{pmatrix}.\]

For this to be zero we have \(a_0 + a_z = a_0 - a_z = 0\), so \(a_0 = a_z = 0\), and \(a_x - ia_y = a_x + ia_y = 0\), so \(a_x = a_y = 0\).

3. The identity and the Pauli matrices are all self-adjoint, so if \(M = a_0 1 + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z\) then

\[
M^* = \bar{a_0} 1 + \bar{a_x} \sigma_x + \bar{a_y} \sigma_y + \bar{a_z} \sigma_z.
\]

Hence \(M = M^*\) iff all the coefficients are real (\(a_0 = \bar{a_0}\) etc.).

1 has trace 2 and the Paulis have trace 0. Hence \(M\) has trace 2\(a_0\) and \(M\) has trace 0 iff \(a_0 = 0\).

Now suppose \(M\) satisfies the first two conditions, so \(M = a_x \sigma_x + a_y \sigma_y + a_z \sigma_z\) with \(a_x\), \(a_y\) and \(a_z\) all real.

Using part (1) we find

\[
M^2 = (a_x^2 + a_y^2 + a_z^2) 1
\]

so \(M^2 = 1\) iff \(a_x^2 + a_y^2 + a_z^2 = 1\).

Thus the Bloch matrices can be identified with points on the sphere.

4. If \(\lambda\) is an eigenvalue and \(|\phi\rangle\) an eigenstate then

\[
|\phi\rangle = M^2 |\phi\rangle = M |\lambda |\phi \rangle = \lambda^2 |\phi \rangle
\]

and it follows (since \(|\phi\rangle \neq 0\)) that \(\lambda^2 = 1\) and so \(\lambda\) is \(\pm 1\). In diagonal form the diagonal entries must therefore be both 1, both \(-1\), or one of each. But to give trace 0 they must be one of each.

5. The eigenspaces for 1 and \(-1\) must have dimension 1 each, so must both be rays. The fix space is the ray generated by any eigenvector for 1, and its orthogonal complement (the subspace of vectors othogonal to everything in the first ray) is generated by any eigenvector for \(-1\). \(M\) is the unique matrix that fixes every vector in the first and negates every vector in the second. Conversely, given any ray the unique \(M\) fixing it and negating its orthogonal complement has matrix \(\sigma_z\) with respect to a basis derived from the two rays and so is a Bloch matrix.