

Exercises 4

Topos + Quantum

For most of these questions we work in a non-commutative C^* -algebra $A = M_n(\mathbb{C})$ of $n \times n$ complex matrices. Associated with it is the n -dimensional Hilbert space \mathbb{C}^n . Its elements, written with Dirac notation $|\phi\rangle$, should be thought of as column vectors (u_i) . “Hilbert space” means just (in this finite-dimensional setting) that it has an inner product. The inner product of $|\phi\rangle$ with the vector $|\psi\rangle = (v_i)$ is

$$\langle\phi|\psi\rangle = \sum_{i=1}^n \bar{u}_i v_i.$$

If $M \in A$ then we write $\langle\phi|M|\psi\rangle$ for the inner product of $|\phi\rangle$ with $M|\psi\rangle$, or (which gives the same result) of $M^*|\phi\rangle$ with $|\psi\rangle$.

FACT: Any commutative C^* -algebra of finite dimension k is isomorphic (as C^* -algebra) to \mathbb{C}^k . The spectrum of the algebra has k elements.

1. In \mathbb{C}^k , define k elements e_i ($1 \leq i \leq k$) by

$$(e_i)_k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

Show that

$$e_i e_j = \begin{cases} e_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$1 = \sum_{i=1}^k e_i$$

Hence each e_i is a *projector*, i.e. self-adjoint and idempotent ($e_i^2 = e_i$).

2. Let C be a commutative sub- C^* -algebra of A , with dimension k . Show that there are k projectors P_i in A such that

$$P_i P_j = \begin{cases} P_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$1 = \sum_{i=1}^k P_i$$

and C is the set of complex linear combinations of the P_i s.

3. Let $C \subseteq C'$ be two commutative sub- C^* -algebras of A , with dimensions k and k' and P_i and P'_j . Show the following.

- (a) There is a function $f : \{1, \dots, k'\} \rightarrow \{1, \dots, k\}$ such that $P_i = \sum_{f(i')=i} P'_{i'}$ for each i .
- (b) For each i' , $f(i')$ is the unique i such that $P'_{i'} P_i = P'_{i'}$. (Thanks to Jorick Mandemaker for pointing out this idea in class.)
- (c) The assignment that takes each C to its corresponding $\{1, \dots, k\}$ is contravariantly functorial on $\mathcal{C}(A)$.¹

This leads to the spectral bundle $\Sigma(A)$, in which the fibre over C is $\{1, \dots, k\}$.

4. Let $|\psi\rangle$ be a non-zero vector in \mathbb{C}^n . For each $C \in \mathcal{C}(A)$, define a probabilistic distribution on $\{1, \dots, k\}$ by

$$\text{prob}(i) = \frac{\langle\psi|P_i|\psi\rangle}{\langle\psi|\psi\rangle}$$

- (a) Show that it is a probability distribution. (All the probabilities are between 0 and 1, and they sum to 1.)
- (b) Show how it is contravariant in C .

¹You might think $\{1, \dots, k\}$ is a set and hence giving a presheaf on $\mathcal{C}(A)$ – and this would be correct in the Imperial approach. But as a finite set it is also a compact Hausdorff space, and this fits the Nijmegen approach. The difference becomes more important in the infinite dimensional case.

This gives a family (one for each $|\psi\rangle$) of cross-sections of the valuation bundle of the spectral bundle, even though the spectral bundle has no cross sections.

Note the contrast.

In the *internal* reasoning (in the topos $\mathcal{T}(A)$), the logic is non-classical but the account of states is classical (the spectrum is a classical state space and the probabilities arise as distributions on that).

In the *external* reasoning, the logic may be classical but the account of states is quantum.

If $M \in A$ is a self-adjoint, corresponding to an observable, then we can find some $C \in \mathcal{C}(A)$ containing M , giving $M = \sum_{i=1}^k \lambda_i P_i$. The λ_i s are the eigenvalues of M . For any non-zero state $|\phi\rangle$, if $P_i|\phi\rangle$ is non-zero then it is an eigenvector of M , with eigenvalue λ_i . There may be repetitions amongst the λ_i s, in which case they need to be grouped together. Now *Born's rule* can be described as follows. Suppose λ is an eigenvalue, and $P = \sum_{\lambda_i=\lambda} P_i$. Then the probability on measuring M of getting result λ is $\frac{\langle\psi|P|\psi\rangle}{\langle\psi|\psi\rangle}$, with the resulting state being $P|\psi\rangle$. This process is independent of choice of C .

Solutions

1. These are obvious, really. $(e_i e_j)_k$ is 1 just when both $(e_i)_k$ and $(e_j)_k$ are both 1, i.e. when $i = k = j$. If $i \neq j$ then this never happens, so $e_i e_j = 0$. If $i = j$ then it tells us that $e_i e_j = e_i$. $(\sum_i e_i)_k = 1$ for all k , so $\sum_i e_i = 1$. The self-adjointness of e_i is because its coordinates are all real.
2. This follows directly from the give Fact, that $C \cong \mathbb{C}^k$. The projectors P_i in C correspond to the projectors e_i in \mathbb{C}^k . Every element of C is a linear combination of the P_i s, because the corresponding fact holds in \mathbb{C}^k – the vector (u_i) is $\sum_i u_i e_i$.

3. (a) Since $P_i \in C'$, it is a linear combination of the $P'_{i'}$ s, $P_i = \sum_{i'=1}^{k'} \alpha_{i'}^{(i)} P'_{i'}$. Using the equations of question (2) we can calculate that $P_i^2 = \sum_{i'=1}^{k'} (\alpha_{i'}^{(i)})^2 P'_{i'}$; then because P_i is idempotent, all the coefficients $\alpha_{i'}^{(i)}$ are either 0 or 1. Since $P_i P_j = 0$ if $i \neq j$, we can also calculate that $\alpha_{i'}^{(i)}$ and $\alpha_{i'}^{(j)}$ cannot both be 1 for the same i' . Hence for each i' there is at most one i for which $\alpha_{i'}^{(i)} = 1$. But since

$$1 = \sum_{i=1}^k P_i = \sum_{i=1}^k \sum_{i'=1}^{k'} \{P_{i'} \mid \alpha_{i'}^{(i)} = 1\},$$

for every i' there must be *exactly* one i with $\alpha_{i'}^{(i)} = 1$. Hence we get a function $f : \{1, \dots, k'\} \rightarrow \{1, \dots, k\}$, $f(i') = i$ iff $\alpha_{i'}^{(i)} = 1$. The result follows.

(b)

$$P'_{i'} P_i = P'_{i'} \sum_{f(j')=i} P'_{j'} = \begin{cases} P'_{i'} & \text{if } f(i') = i \\ 0 & \text{otherwise} \end{cases}$$

Hence $f(i')$ is the unique i such that $P'_{i'} P_i = P'_{i'}$.

- (c) Suppose we have $C \subseteq C' \subseteq C''$ with all the corresponding notation including $f : \{1, \dots, k'\} \rightarrow \{1, \dots, k\}$ and $f' : \{1, \dots, k''\} \rightarrow \{1, \dots, k'\}$. If $f'(i'') = i'$ and $f(i') = i$, then

$$P''_{i''} P_i = P''_{i''} P'_{i'} P_i = P''_{i''} P'_{i'} = P''_{i''}$$

and so i'' map to i under the function corresponding to $C \subseteq C''$. It follows that the assignment preserves composition. Preservation of identities is obvious.

4. (a)

$$\begin{aligned} \sum_{i=1}^k \frac{\langle \psi | P_i | \psi \rangle}{\langle \psi | \psi \rangle} &= \frac{1}{\langle \psi | \psi \rangle} \sum_{i=1}^k \langle \psi | P_i | \psi \rangle = \frac{1}{\langle \psi | \psi \rangle} \langle \psi | \sum_{i=1}^k P_i | \psi \rangle \\ &= \frac{1}{\langle \psi | \psi \rangle} \langle \psi | 1 | \psi \rangle = \frac{\langle \psi | \psi \rangle}{\langle \psi | \psi \rangle} = 1. \end{aligned}$$

We must still show each $\langle \psi | P_i | \psi \rangle \geq 0$. (Note that $\langle \psi | \psi \rangle > 0$, since it is a sum of squared magnitudes and $|\psi\rangle \neq 0$.) A simple proof relies on the fact that P_i is a projector, so the inner product of $P_i |\psi\rangle$ with itself (which is again a sum of squared magnitudes, so non-negative) is

$$\langle \psi | P_i^* P_i | \psi \rangle = \langle \psi | P_i P_i | \psi \rangle = \langle \psi | P_i | \psi \rangle.$$

More deeply, the eigenvalues of P_i are all 0 or 1, which are non-negative reals. If we use a diagonalizing basis then $\langle \psi | P_i | \psi \rangle$ is a sum of squared magnitudes weighted by the eigenvalues (the diagonal elements), and hence non-negative.

- (b) Suppose $C \subseteq C'$ in $\mathcal{C}(A)$, with notation as in question (3). Suppose we have a probability distribution $(p_{i'})_{i'=1}^{k'}$ on $\{1, \dots, k'\}$, with $0 \leq p_{i'}$ and $\sum_{i'} p_{i'} = 1$. Then we can define a probability distribution $(p_i)_{i=1}^k$ on $\{1, \dots, k\}$ by $p_i = \sum_{f(i')=i} p'_{i'}$. This extends the map f between the spectra into a map between their valuation spaces.

In the present situation we have

$$\begin{aligned} p_i &= \frac{\langle \psi | P_i | \psi \rangle}{\langle \psi | \psi \rangle} \\ p'_{i'} &= \frac{\langle \psi | P'_{i'} | \psi \rangle}{\langle \psi | \psi \rangle} \end{aligned}$$

Then

$$\begin{aligned} p_i &= \frac{\langle \psi | P_i | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{1}{\langle \psi | \psi \rangle} \langle \psi | \sum_{f(i')=i} P'_{i'} | \psi \rangle \\ &= \sum_{f(i')=i} \frac{\langle \psi | P'_{i'} | \psi \rangle}{\langle \psi | \psi \rangle} = \sum_{f(i')=i} p'_{i'}. \end{aligned}$$

Hence the distributions we have defined from $|\psi\rangle$ form a cross-section of the valuation bundle.