Quantum Topos Theory

Toposes, Quantum Theory, & how they fit together

1. Topology via test functions: What is continuity?
2. Toposes
3. Quantum Theory
4. = 2 + 3

Steve Vickers

Topology via test functions

- Various test spaces
- Giving different notions of topology
- \( \mathbb{C} \) or \( \mathbb{R} \)
- Sierpinski $\star$
- Stone spaces
- Gelfand duality with C*-algebras
- \( \rightarrow \) Quantum
- Sets
- Toposes

What is continuity? - Informally

1. No jumps or tears => no discontinuities
   - e.g. discontinuous function
   - Discontinuous deformation of rubber sheet

What is continuity? - Informally

2. (Computational)
   - Finite piece of output requires only finite information from input
   - e.g. bitstream transformers
   - Continuous: \( n \) th bit of output = \( 2^n \) th bit of input
   - Not continuous:
     \[ n \text{ th bit of output} = \begin{cases} 1 & \text{if all of input is 1s} \\ 0 & \text{otherwise} \end{cases} \]
1 & 2 related

1. No jumps or tears (discontinuities)
2. (Computational)
   Finite piece of output requires only
   finite information from input.

If observing a
real number (e.g., physical measurements):
never know it exactly
only to some approximation

Finite information = approximation within some
for reals error bound

Approximate knowledge

If f continuous:
- don’t have to know x exactly
- there is some “neighbourhood” of
  approximations x’ to x with all f(x’)
  having that same “quality”

Knowing x ∈ that neighbourhood is “finite
information about input”

What is continuity? - Informally

3. Logic of finite observations

Logical negation problematic
e.g. “bit stream has a 0” - observable
negation “bit stream never has a 0” - infinite info.

Classical logic
   logic of truth
   true    false

Geometric logic
   logic of finite observation
   ascertained => you know it's true
   not ascertained => you don't know

Continuity as logic

Continuous function:
finite amounts of input
→ finite amounts of output

Geometric logic respects notions of
“finite observation” - inherently continuous

Classical logic allows discontinuity

Slogan: Continuity is geometricity

We’ll see some of this with the toposes
What is continuity? - Formally

Use **Topological space**

Traditional definition

T. Haudsdorff → S. Kuratowski

1914, 1922

- set \( X \)
- set \( \mathcal{O}_X \) of subsets of \( X \)
  - \( \emptyset, X \) are open
  - \( U, V \) open \( \Rightarrow U \cap V \) open
  - \( U \) open \((x \in I) \Rightarrow U; x; \) open

Continuity \( f: X \to Y \) continuous

if \( \forall \varepsilon > 0 \Rightarrow f^{-1}(\varepsilon) \subseteq \mathcal{O}_X \)

e.g., real line \( \mathbb{R} \)

\[ U \text{ open} \text{ if a union of open intervals} \]

\[ (x_1, y) = \{ y \mid x_1 < y \leq y \} \]

Open intervals a base of opens

\[ \forall x \in U \; \exists \varepsilon > 0 \; (x - \varepsilon, x + \varepsilon) \subseteq U \]

\[ u \]

\[ U \text{ doesn't contain its boundaries points} \]

\( f: X \to \mathbb{R} \) continuous?

- \( x \in \mathbb{R} \)

\[ (f(x) - \varepsilon, f(x) + \varepsilon) \text{ open} \]

\[ \Rightarrow f^{-1}(f(x) - \varepsilon, f(x) + \varepsilon) \text{ open, contains } x \]

\[ x - \delta < x < x + \delta \Rightarrow f(x) - \varepsilon < f(x) < f(x) + \varepsilon \]

\[ \text{Jump \Rightarrow not continuous} \]

Approximate knowledge

If \( f \) continuous:

- To know \( f(x) \) has "some quality"
- Don't have to know \( x \) exactly
- There's some neighbourhood of approximations \( x' \) to \( x \) with all \( f(x') \) having that same "quality"

Knowing \( x \in U \) that neighbourhood is "false information about input"
"Approximations" in program semantics

Opens ≈ finite amount of observable information

E.g., bit streams $2^\mathbb{N}$

Open property: ascertained by observing finite portion of stream

Computable $\Rightarrow$ every action based on finite information

$\Rightarrow$ continuous

Sierpinski as domain of truth values

Think: logic of observation

$T$ - "ascertained" - now certain property holds

$\bot$ - "not ascertained" - don't know

For classical truth values true/false use $2 = \{0, 1\}$, all subsets open

False $\Rightarrow$ true

Sierpinski as test space

What are continuous maps $f: X \to \mathcal{P}$?

$\mathcal{P}$ open in $\mathcal{P}$

$\Rightarrow f^{-1}(\mathcal{P})$ open in $X$

But - if you know $f^{-1}(\mathcal{P})$

you know all of $f$

$\Rightarrow$ bijection

Continuous maps $\approx$ Opens in $X$

$X \to \mathcal{P}$

E.g., Sierpinski space $\mathcal{P}$

Two points $\bot \top$

Opens $\emptyset, \{\top\}, \mathcal{P}$

Specialization $\Rightarrow$ specializes $x$

$\forall y \in \mathcal{P}$ if every open containing $x$ also contains $y$ more specific information for $y$

$\bot \in \mathcal{P}$ in $\bot$

in $\mathbb{R}$, $x \in \mathcal{P}$ only when $x = y$ $\Rightarrow$ $x \in \mathbb{R}$
Continuity using $\mathbb{R}$

Topology is the same as saying which maps $X \to \mathbb{R}$ are continuous.

Continuity $f : X \to Y$

If $V \subseteq Y$ corresponds to $g : Y \to \mathbb{R}$

then $f^{-1}(V) \subseteq X$.

$\therefore f$ continuous if $g$ continuous then so is $g \circ f$.

Continuous operations on $\mathbb{R}$

$\land : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\land (p,q) = \begin{cases} T & \text{if } p = q = T \\ \bot & \text{otherwise} \end{cases}$

$\land^{-1}(\{T\}) = \{T\} \times \{T\}$

Continuous? $\land^{-1}(\{T\}) = \{T\} \times \{T\}$

To topologize $X \times Y$:

- If $U \subseteq X$, $V \subseteq Y$ then $U \times V$ open in $X \times Y$.
- $U \cap V$ also open.

$\therefore \land$ is continuous.

Discontinuous operations on $\mathbb{R}$

$\lor : \mathbb{R} \to \mathbb{R}$

$\lor (\top, \bot) = \top$.

Continuous? $\lor^{-1}(\{T\}) = \{\top\} \times \{\top\}$

$\lor^{-1}(\{T\}) = \{\top\} \times \{\top\}$

Definition of opens for infinite products allows only finitely many of these to be different from $\bot$. 

$\lor^{-1}(\{T\}) = \{\top\} \times \{\top\} \times \cdots \times \{\top\}$
Definition of topology matches continuous operations on $\emptyset$.

Opens $\emptyset$, Maps $X \rightarrow \emptyset$:

- $\forall u, v 
\begin{align*}
\langle f_u, f_v \rangle : X &\rightarrow \emptyset \\
\langle f_u, f_v \rangle (x) &= (f_u(x), f_v(x)) \\
\cap \langle f_u, f_v \rangle (x) &\Rightarrow x \in U \text{ and } x \in V
\end{align*}$

Idea: topology follows algebra of test space

- Test space $T$ has operations $\cup$, $\cap$.
- To topologize $X$, specify continuous test maps $X \rightarrow T$.
- These should form algebra with same operations as $T$ (on maps: operate argument-wise).
- $f : X \rightarrow Y$ is continuous if $g : Y \rightarrow T$ continuous $\Rightarrow$ so is $g \circ f$.

Definition of topology matches continuous operations on $\emptyset$.

Opens $\emptyset$, Maps $X \rightarrow \emptyset$:

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\cap \langle f_u, f_v \rangle (x) &\Rightarrow x \in U \text{ and } x \in V
\end{align*}$

Other test spaces?

1. $\mathbb{R}$, discrete topology:
   - all subsets are open
   - Continuous map $X \rightarrow \mathbb{R}$ is open determined by $f'(\mathbb{R})$ - open
   - $f''(\mathbb{R})$ - also open
   - $f''(\mathbb{R})$ is closed
   - $f''(\mathbb{R})$ is clopen - open and closed
   - Continuous maps $\sim$ clopens in $X$
Operations on 2

Constants 0, 1
1 \land 1 = 2
1 \lor 1 = 2

\lor, \land : 2 \times 2 \rightarrow 2

2 is a Boolean algebra

\Rightarrow so is set of clopens in X

Continuity using 2 ?

\text{f continuous } \iff \text{g \gamma \rightarrow Y continuous, then so is g \rightarrow f}

\text{No good if too few clopens}

\text{e.g. R : only two clopens } \emptyset, R

\text{if } U \subset R \text{ is open & has a boundary point :}

\text{a \in U}

\text{a \in R-U}

\text{a on boundary of both } U \text{ and } R-U

\text{\Rightarrow R-U not open}

\text{\Rightarrow U not closed}

OK if every open in Y is union of clopens.

Continuity using 2 ?

\text{OK if every open in Y is union of clopens.}

\text{e.g. bitstreams } 2^\mathbb{N}

\text{U clopen if property of only finitely many bits in stream}

\text{e.g. } U = \{ s \in 2^\mathbb{N} \mid s_2 = 0 \text{ and either } s_3 \text{ or } s_4 \text{ is 1} \}

Stone spaces

\text{Special spaces : clopens are enough}

\text{Use 2 as test space to define topology}

Won't give def. here
Other test spaces: $\mathbb{C}^*$ (complex numbers)

No good for $\mathbb{R}$

- $f: \mathbb{R} \to \mathbb{C}$ must preserve $\ell$
  - $f(\ell) = f(\ell)$
- But $\ell$ in $\mathbb{C}$ is =
  - $f(\ell) = f(\ell)$
- No map can distinguish $1$ from $i$

Continuous operations on $\mathbb{C}$
- $\cdot, -, \times$
- Complex conjugation

$\mathbb{C}^*$-algebras have this structure
(+ a little more)

Spaces from algebras

From space to algebra
- Fix $X$
- Let $C^*_T(X)$ comprise the specified continuous test maps $X \to T$
- $C^*_T(X)$ has same algebraic structure (operations, laws) as $T$
- Also has "constant" map $X \to T$ for each $a \in T$
- Get homomorphism $T \to C^*_T(X)$ preserves operations
- Continuous map $f: X \to T$
  - Gives homomorphism $C^*_T(f): C^*_T(X) \to C^*_T(T)$
From algebra to space?

Each $x \in X$ gives homomorphism $e_x : T \to \mathbb{C}$, "evaluate at $x$".

Given algebra $A$, with homomorphism $T \to A$.

Think of homomorphisms $A \to T$ as abstract points $pt(A)$.

Topologize $pt(A)$? $pt(A) \times A \to T$.

Each $a$ gives map $p(A) \to T$.
- These are to be the continuous ones.

Space to algebra & back again?

Concrete & abstract not quite the same.

$X \to pt(e_x : X)$
- Can have two concretes $\to$ same abstract.
  e.g., $1, T \in \mathbb{S}$ when test space $2$ or $\mathbb{C}$.
  $2, \mathbb{C}$ can’t distinguish $1$ for $T$.
- Can have abstracts not induced by any concretes.

Idea

- Algebras are best uniform way to describe spaces.
- Concrete points are just labels for abstract point.
- Test space $T$ as algebra = 1 point space.
- Continuous maps are algebra homomorphisms backwards.
  - Continuous map $f : X \to T$ gives homomorphism $f \circ e_x : C \to C$.

Examples 1: $T = \mathbb{S}$

Algebra = frame = $\land \lor$
- Complete lattice + a distributive law
- Get locale theory

Broadly in line with topology.
Think: locales are spaces where topology lives.
- Topological space = locale + labels for points.
Examples ②: \( T = 2 \)

Algebra = Boolean algebra \( \land, \lor, \neg \)

Cannot deal with
  - non-trivial specialization order \( \text{e.g.} \subset \)
  - connectedness \( \text{e.g.} \mathbb{R} \)

Exact match with some topological spaces:
  - Stone spaces
    - Stone's Representation Theorem

Examples ③: \( T = \mathbb{C} \)

Algebra = \( C^* \)-algebra

Cannot deal with
  - non-trivial specialization order \( \text{e.g.} \subset \)
  - connectedness

Exact match with some topological spaces:
  - compact Hausdorff spaces
    - Gelfand–Naimark duality

Examples ④: \( T = \text{Sets} \)

Algebra = Grothendieck topos

Next lecture:
  - for ordinary topology specify opens = continuous maps to \( \$ \)
  - for topos specify sheaves = continuous maps to \( \text{Sets} \)