Midland Graduate School 2010
Quantum Topos Theory

Quantum + Topos

- Imperial = Isham + ① Butterfield ② Döring
- Nijmegen = Heunen, Landsman, Spitters
- etc.

Steve Vickers

Start with: non-commutative C^*-algebra A

- to describe physical system under consideration

Measurement = self-adjoint $M \in A$ ∴ $M = M^*$

Evolution of system < computation, time evolution

= unitary $M \in A$ ∴ $M^* = M^{-1}$

Illustrative example: $A = M_n(\mathbb{C})$ Hilbert space is $\mathbb{C}^n$

Finite dimensional - good enough for quantum computation.

e.g. m Qbits — $n = 2^m$

Infinite dimensions - more complicated

- needed for some physics situations
- Imperial restrict to von Neumann algebras

better adapted to measure theory

less to topology ?

I'll illustrate with finite dimensions

- though infinite dim. more interesting topologically

Spectrum of operator M

$n \times n$ matrices still

- If M normal then diagonalizable

$MM^* = M^*M$

Matrix $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$ w.r.t. suitable basis of eigenstates $|\phi_i\rangle$

k need not be n - may have eigenvalues repeated

$M|\phi_i\rangle = \lambda_i|\phi_i\rangle$

If λ_i repeated, need more than one $|\phi_i\rangle$

Normal includes -

self-adjoint : all λ_is are real

$M^* = M^{-1}$ ∘ unitary : all λ_is are unit length in $\mathbb{C}$

Projectors

If M normal then diagonalizable

Matrix $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$ w.r.t. suitable basis of eigenstates $|\phi_i\rangle$

Use same basis of $|\phi_i\rangle$s

Projector = selfadjoint idempotent $P^2 = P$

Projector P_i has matrix got by replacing:
λ_i by 1
other λ_js by 0

e.g. $M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- $M = \sum \lambda_i P_i$
- M & P_is all commute
- $P_i P_j = \begin{cases} P_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
- $\sum P_i = 1$

Commutative subalgebra

Let $C = \{$linear combinations of P_is$\}$
 - commutative sub-C^*algebra of $M_n(\mathbb{C})$

Spectrum of $C = k =$ number of eigenvalues

Measuring M: eigenvalues used as labels to show which eigenspace end up in

For one M get well-defined state space.

Can extend to any commuting family of M's.

Non-commuting operators

Commuting operators $\Rightarrow$ can measure simultaneously
- find basis that diagonalizes both
- get commutative subalgebra to contain both
- measurement $\to$ eigenspace for both

Non-commuting operators $\Rightarrow$ can't.

e.g. Heisenberg's uncertainty principle
Can't simultaneously know position and momentum
Lower bound on product of uncertainties

Commutative subalgebras = classical points of view

Isham

Commutative subalgebra
$\Longrightarrow$ classical state space

spectrum

Doesn't extend to cover all measurables

Kochen-Specker $\Rightarrow$ can't

Can't reconcile all the classical points of view

Isham-Butterfield: Use internal topos logic to deal with classical points of view all together

Externally? Get probabilistic info.

Basic plan

- Start with a non-commutative C^*-algebra A to describe physical system
- Let $\mathcal{C}(A) = \{$commutative sub-C^*-algebras of $A\}$
  - partial order under $\subseteq$
- Construct topos $\mathcal{T}(A)$ whose points include elements $C \in \mathcal{C}(A)$
- Construct spectral bundle $\Sigma(A)$ fibrewise

General result

Exercise 2

$\begin{array}{c} Y \\ p \downarrow \\ X \end{array}$ a local homeomorphism,

$x \sqsubseteq x'$ in X

specialization order: $\forall$ open U $x \in U \to x' \in U$

Get function $p^{-1}(\{x\}) \longrightarrow p^{-1}(\{x'\})$

$y \longmapsto$ unique y' s.t. $y \sqsubseteq y'$

Fibres vary covariantly with base point.

- for local homeomorphisms, not for bundles in general

General result

exercise 2

If $(P, \leq)$ a poset:

- ideal is $I \subseteq P$ that is down-closed, inhabited, & any two $x, y \in I$ have upper bound in I
- principal ideal is $\downarrow x = \{y \mid y \leq x\}$ for $x \in I$.
- $\mathrm{Idl}(P) = \{$ideals$\}$, $\downarrow : P \to \mathrm{Idl}(P)$ order embedding
- Scott topology on $\mathrm{Idl}(P) \cong \{$up closed subsets of $P\}$
- sheaves on $\mathrm{Idl}(P) \simeq$ functors $P \longrightarrow$ Sets

"continuity" of map $\mathrm{Idl}(P) \to$ Set $\Rightarrow$ takes directed joins to directed colimits $\Rightarrow$ determined by action on principal ideals

cf. Scott continuity

Imperial version

$\mathcal{T}(A)$ = category of presheaves over $\mathcal{C}(A)$

- contravariant functors $(\mathcal{C}(A), \subseteq) \to$ Sets

$\simeq$ sheaves on $\mathrm{Idl}(\mathcal{C}(A)^{op})$

describes all the points – usually enough to consider principal ideals from ?

Spectral bundle $\Sigma(A)$ is local homeomorphism (sheaf)

Fibre over $C \in \mathcal{C}(A)$ = set of points of spectrum of C

commutative sub-C^*-algebra

$pt(\mathrm{Spec}(C))$

Variance

"coarse-graining"

$C \subseteq C'$ in $\mathcal{C}(A)$

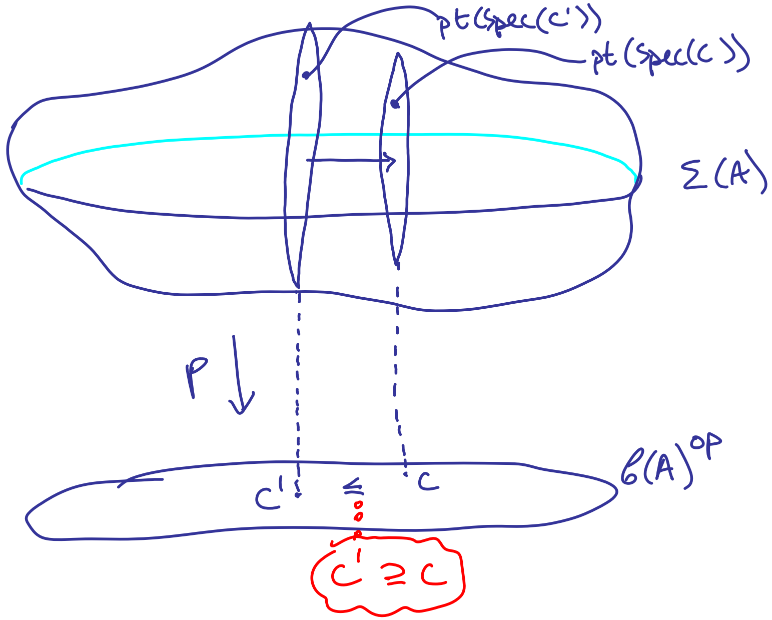
spectral duality $\Rightarrow$ $\mathrm{Spec}(C) \longleftarrow \mathrm{Spec}(C')$

Spec is contravariant over $\mathcal{C}(A)$

covariant over $\mathcal{C}(A)^{op}$

$\mathrm{pt}(\mathrm{Spec})$ is sheaf over $\mathrm{Idl}(\mathcal{C}(A)^{op})$

(presheaf over $\mathcal{C}(A)$)



Kochen-Specker

Cannot choose $\sigma(C) \in \mathrm{Spec}(C)$ for every $C \in \mathcal{C}(A)$, consistent with ordering

σ would be cross-section of spectral bundle

$\sigma(C) \in p^{-1}(\{C\})$ - $p \circ \sigma = \mathrm{Id}$

$\therefore$ Spectral bundle has no cross-sections

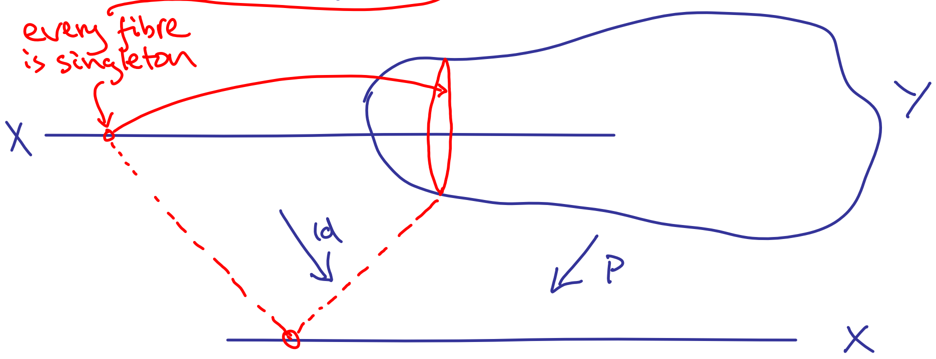
for dimension of $\mathcal{H} \geq 3$

in internal mathematics of sheaves

Global elements = cross-sections

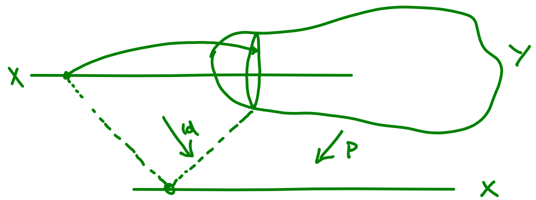
Global element of sheaf Y

= map from constant-1 sheaf to Y



Global elements

= cross-sections



For each x, get $\sigma(x) \in p^{-1}(\{x\})$

$\sigma : X \longrightarrow Y$ continuous

$p \circ \sigma = \mathrm{Id}_X$

$\therefore$ Kochen-Specker says:

spectral sheaf has no global elements

No external classical states.

Nijmegen version

Different variance

$\mathcal{J}(A)$ = category of presheaves over $\mathcal{C}(A)^{op}$

CO - ~~contravariant~~ functors $(\mathcal{C}(A), \subseteq) \to \mathrm{Sets}$

$\simeq$ sheaves on $\underbrace{\mathrm{Idl}(\mathcal{C}(A)^{\cancel{op}})}$

describes all the points - usually enough to consider principal ideals from?

Spectral bundle $\Sigma(A)$ is ~~local homeomorphism (sheaf)~~ got by Gelfand-Naimark

Fibre over $C \in \mathcal{C}(A)$ = ~~set of points of~~ spectrum of C

commutative sub-C*-algebra

$(\cancel{pt}(\mathrm{Spec}(C)))$

Nijmegen version

- "Tautologous" sheaf $\underline{A}$: stalk at C is C
- Internal commutative C*-algebra in $\mathcal{J}(A)$
  because stalks commutative
- Apply Banaschewski-Mulvey
  Gelfand-Naimark in topos
- Get internal compact completely regular locale
- $\Longrightarrow$ external bundle

  $\parallel$

  spectral bundle $\Sigma(A)$
- As before, no cross sections

Variance

- $\underline{A}$: fibre ($= C$) varies covariantly with C
- $\Sigma(A)$: fibre ($= \mathrm{Spec}(C)$) varies contravariantly with C

$\therefore$ cannot be local homeomorphism for Nijmegen topos (over $\mathrm{Idl}(\mathcal{C}(A))$)

Shouldn't be a problem -

local homeomorphism ~ fibrewise discrete

$\mathrm{Spec}(C)$ ~ fibrewise compact Hausdorff.

Probability

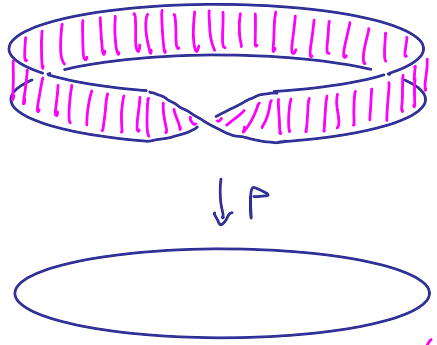
Nijmegen

- $\Sigma(A)$ internal locale
- Construct $\mho(\Sigma(A))$
- $\mho(\Sigma(A))$ does have cross-sections
- Can get external distributions of classical states, even though no external classical states.

Valuation space $\mho(Y)$
- space of valuations (regular measures)
  $\mu : \Omega Y \longrightarrow \mathbb{R}$ - Scott continuous
  $\mu(U) + \mu(V) = \mu(U \wedge V) + \mu(U \vee V)$
  $\mu(\emptyset) = 0$ + (optionally) e.g. $\mu(Y) = 1$
- $\approx$ probabilistic distribution over Y
- Can be constructed internally in any $\mathcal{E}$
- Works fibrewise on bundles.

Non-quantum example of principle

Bundle space
Y = edges of Möbius band



$\mho(P\downarrow^{Y}_{X})$ is whole of Möbius band - prob. distribution of two edges

Base space
X = circle

$P\downarrow^{Y}_{X}$ has no cross-sections

$\mho(P\downarrow^{Y}_{X})$ has cross-sections

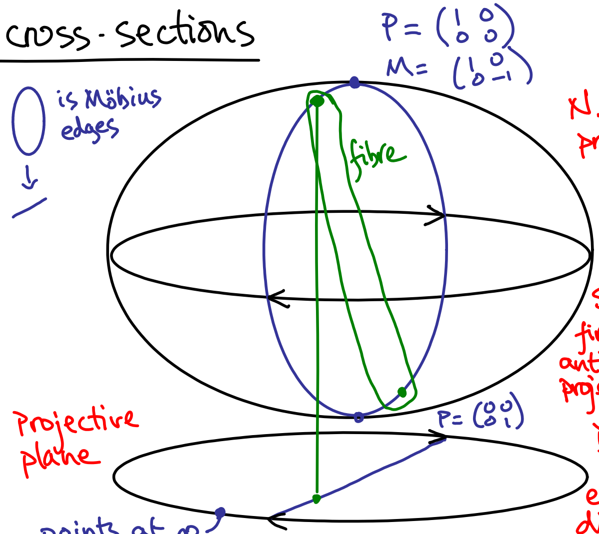
Beyond presheaves?

Topology on $\mathrm{Idl}(\mathcal{C}(A)^{op})$ determined by order
then need sheaves
But $\mathcal{C}(A)$ may have its own topology.
e.g. $A = M_2(\mathbb{C})$
Commutative subalgebras? Dim = 1 or 2
matrices $\propto 1$
Dim 2: determined by two projectors $P, 1-P$
Projectors ~ points on Bloch sphere
$P, 1-P$ are antipodal
They define same subalgebra
$\therefore \mathcal{C}(A)$ = 2-sphere S^2 / identify antipodes = Real projective plane $\mathbb{R}P^2$
Spectral bundle is $S^2 \to \mathbb{R}P^2$. Two points in each fibre $(P, 1-P)$

Exercise 3
$P \mapsto M = 2P - 1$
$1 - P \mapsto -M$

No cross-sections

is Möbius edges



N. hemisphere projects down to disc

S. hemisphere: first take antipode, then project down

Projective plane

Disc, identify ends of diameters

Kochen-Specker?

Doesn't apply for $M_2(\mathbb{C})$
- combinatorial result, relies on interplay between different subalgebra dimensions
- doesn't rely on continuity

But even for $M_2(\mathbb{C})$, continuity rules out cross-sections

Conclusions

Hoping:
- test out geometric logic by applying it to the topos approach to quantum
- use benefits of geometricity to simplify it & extend from presheaves to sheaves
- cast a new light on the logic required for quantum theory